Model Theory for abelian $C^*$-algebras

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(Joint with Christopher J. Eagle, University of Toronto)
• Background - C*-algebras
• Model Theory and saturation
• The commutative 0-dimensional case
Definition

A $C^*$-algebra is a complex Banach space $A$ with a multiplication $\cdot$ and an involution $^*$ satisfying:

- $\|xy\| \leq \|x\| \|y\|$, 
- $(x + y)^* = x^* + y^*$ and $(xy)^* = y^*x^*$, 
- $(\lambda x)^* = \overline{\lambda} x^*$
- $\|x\| = \|x^*\|$ and ($C^*$-equality) $\|xx^*\| = \|x\|^2$. 

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Example

- $M_n(\mathbb{C})$, for every $n$,
- $\mathcal{B}(H)$, the algebra of bounded linear operators $T : H \rightarrow H$, where $H$ is a complex Hilbert space (e.g. $\ell^2(\mathbb{N})$),
- $\ast$-closed Banach subalgebras of $\mathcal{B}(H)$.
- $C(X) = \{ f : X \rightarrow \mathbb{C} \mid f \text{ is continuous} \}$ for $X$ compact Hausdorff, with pointwise operations.
- sums, tensor products and products of the above.

Theorem (Gelfand-Naimark-Segal)

- Every unital commutative $C^*$-algebra is isomorphic to $C(X)$, for some $X$ compact Hausdorff.
- Every $C^*$-algebra $A$ is isomorphic to a subalgebra of $\mathcal{B}(H)$, for some $H$. If $A$ is separable, $H$ can be chosen to be separable too.
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Definition

- Let $A$ a $C^*$-algebra and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$
- $\ell^\infty(A) = \{(a_n) \mid a_n \in A, \sup_n \|a_n\| < \infty\}$
- $c_\mathcal{U}(A) = \{(a_n) \mid a_n \in A, \lim_{n \to \mathcal{U}} \|a_n\| = 0\}$ is an ideal in $\ell^\infty(A)$.
- The ultrapower is defined as $A^\mathcal{U} = \ell^\infty(A)/c_\mathcal{U}(A)$.

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Unfortunately first-order logic as we know it is not the right logic for studying $C^*$-algebra

Instead we will use a logic with truth values in $\mathbb{R}$, developed for the study of metric structures (Ben Yaacov - Berenstein - Henson - Usvyatsov) in a version adapted for $C^*$-algebras (Farah - Hart - Sherman).

It was proved that this is the right logic (there is a Lindstrom-like Theorem) if one wants basic model theory (Lowenheim-Skolem $\uparrow$ and $\downarrow$, compactness, unions of elementary chains, etc).
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Atomic formulas are of the form $\|P(\bar{x})\|$ where $P(\bar{x})$ is a $^*$-polynomial.

If $\phi_1(\bar{x}), \ldots, \phi_n(\bar{x})$ are formulas, and $f : \mathbb{R}^n \to \mathbb{R}$ is uniformly continuous, then $f(\phi_1(\bar{x}), \ldots, \phi_n(\bar{x}))$ is a formula.

If $\phi$ is a formula and $R \in \mathbb{R}^+$, $\sup_{\|\bar{x}\| \leq R} \phi(\bar{x})$ and $\inf_{\|\bar{x}\| \leq R} \phi(\bar{x})$ are formulas.

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If $\phi(x_1, \ldots, x_n)$ is a formula, $A$ is a $C^*$-algebra and $a_1, \ldots, a_n$ we can obtain the value $\phi^A(a_1, \ldots, a_n) \in \mathbb{R}$ by substituting $a_i$ for $x_i$ and using operations and norm of $A$.

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Th(A) the set of sentences that are "true" in A.

\[ Th(A) = \{ \sigma \mid A \models \sigma \} = \{ \sigma \mid \sigma^A = 0 \} \]

A and B are elementary equivalent (\( A \equiv B \)) if \( Th(A) = Th(B) \).

**Theorem (Keisler-Shelah)**

- \( A \equiv A^U \) for every ultrafilter \( U \)
- \( A \equiv B \) if and only if there is an ultrafilter \( U \) such that \( A^U \cong B^U \). If the Continuum Hypothesis holds \( U \) is an ultrafilter on \( \mathbb{N} \), otherwise a larger index set may be needed.
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(\sup_{\|x\| \leq 1} \sup_{\|y\| \leq 1} \|xy - yx\|)^A = 0
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*if and only if* \(A\) *is abelian.*

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(\inf_{\|x\| = 1} \|x^2\|)^A = 0
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*if and only if* \(A\) *is not abelian.*

We focus on abelian \(C^*\)-algebras. Then \(A \cong C(X)\). The following topological properties can be detected by the theory:

- being connected or disconnected
- having Lebesgue covering dimension \(=, \leq\) or \(\geq\) than \(n\)
- being indecomposable, hereditary indecomposable
- ..and much more!
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We focus on the 0-dimensional case. Every $X$ is compact Hausdorff. A topological space is 0-dimensional if every open set contains a nontrivial clopen (closed+open). In this case the set of clopens $CL(X)$ is a Boolean algebra. It makes sense to talk about $CL(X) \equiv CL(Y)$ as Boolean algebras, in classical discrete model theory.

**Theorem (Eagle-V.)**

$CL(X) \equiv CL(Y)$ if and only if $C(X) \equiv C(Y)$.

**Theorem (??)**

There are only $\aleph_0$-many theories of Boolean algebras.

**Corollary**

There are $\aleph_0$-many theories of abelian $C^*$-algebras of dimension 0. Moreover

- $C(\alpha + 1) \equiv C(\beta + 1) \equiv C(\beta \mathbb{N})$ for every ordinal $\alpha, \beta$.
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*There are $\aleph_0$-many theories of abelian $C^*$-algebras of dimension 0.* Moreover

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**Theorem**

- *For every* $n > 0$, *there are continuum many theories for* $n$-*dimensional abelian* $C^*$-*algebras.*

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Definition

- Let $A$ be a $C^*$-algebra. Let $\Phi$ be a set of formulas with parameters from $A$. A $\Phi$-type is a set of conditions of the form $\phi(\bar{x}) \in K$ where $\phi \in \Phi$ and $K \subseteq \mathbb{R}$ is compact.
- A $\Phi$-type $\Sigma$ is consistent if for every $\epsilon > 0$ and a finite $\Delta \subseteq \Sigma$ there is $\bar{a} \in A$ such that $d(\phi(\bar{a}), K) < \epsilon$ for every $\phi \in \Delta$.
- A $\Phi$-type is realized if there is $\bar{a} \in A$ such that $\phi(\bar{a}) \in K$ for every $\phi \in \Sigma$.
- $A$ is said countably $\Phi$-saturated if every consistent countable $\Phi$-type is realized.

Three degrees of saturation are important: 1-degree ($\Phi$ is the set of formulas of the form $\|P\|$, with $P$ a $*$-polynomial of degree one), quantifier free ($\Phi$ is the set of quantifier free formulas), and full.
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If $A$ is a $C^*$-algebra, $A^U$ is **full** saturated for every nonprincipal $U$.

- If $A$ is separable, $\ell^\infty(A)/c_0(A)$ is **full** saturated [Farah-Shelah]
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**Theorem (Eagle - V.)**

*If $C(X)$ is countably 1-degree saturated:*

- $X$ is not metrizable
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In particular the theory of $C(2^\mathbb{N})$ has quantifier elimination.

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