Model Theory for abelian C^*-algebras

Alessandro Vignati
York University, Toronto

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(Joint with Christopher J. Eagle, University of Toronto)

- Background - C^*-algebras
- Model Theory and saturation
- The commutative 0-dimensional case

Definition

A C^*-algebra is a complex Banach space A with a multiplication $\cdot$ and an involution * satisfying:

- $\|xy\| \leq \|x\| \, \|y\|$,
- $(x+y)^* = x^* + y^*$ and $(xy)^* = y^*x^*$,
- $(\lambda x)^* = \overline{\lambda} x^*$
- $\|x\| = \|x^*\|$ and (C^*-equality) $\|xx^*\| = \|x\|^2$.

Example

- $M_n(\mathbb{C})$, *for every* n,
- $\mathcal{B}(H)$, *the algebra of bounded linear operators* $T : H \to H$, *where* H *is a complex Hilbert space (e.g.* $\ell^2(\mathbb{N})$*),*
- **-closed Banach subalgebras of* $\mathcal{B}(H)$.
- $C(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}$ *for* X *compact Hausdorff, with pointwise operations.*
- *sums, tensor products and products of the above.*

Theorem (Gelfand-Naimark-Segal)

- *Every unital commutative* C^**-algebra is isomorphic to* $C(X)$*, for some* X *compact Hausdorff.*
- *Every* C^**-algebra* A *is isomorphic to a subalgebra of* $\mathcal{B}(H)$*, for some* H*.If* A *is separable,* H *can be chosen to be separable too.*

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Definition

- Let A a C^*-algebra and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$
- $\ell^\infty(A) = \{(a_n) \mid a_n \in A, \sup_n \|a_n\| < \infty\}$
- $c_{\mathcal{U}}(A) = \{(a_n) \mid a_n \in A, \lim_{n\to\mathcal{U}} \|a_n\| = 0\}$ is an ideal in $\ell^\infty(A)$.
- The ultrapower is defined as $A^{\mathcal{U}} = \ell^\infty(A)/c_{\mathcal{U}}(A)$.

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- Unfortunately first-order logic as we know it is not the right logic for studying C^*-algebra
- Instead we will use a logic with truth values in $\mathbb{R}$, developed for the study of metric structures (Ben Yaacov - Berenstein - Henson - Usvyatsov)in a version adapted for C^*-algebras (Farah - Hart - Sherman).
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- **Atomic formulas** are of the form $\|P(\overline{x})\|$ where $P(\overline{x})$ is a $*$-polynomial.
- If $\phi_1(\overline{x}), \ldots, \phi_n(\overline{x})$ are formulas, and $f\colon \mathbb{R}^n \to \mathbb{R}$ is uniformly continuous, then $f(\phi_1(\overline{x}), \ldots, \phi_n(\overline{x}))$ is a formula.
- If ϕ is a formula and $R \in \mathbb{R}^+$, $\sup_{\|x\| \leq R} \phi(x)$ and $\inf_{\|x\| \leq R} \phi(x)$ are formulas.

Note that there is no negation, nor implication.

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- If $\phi(x_1, \ldots, x_n)$ is a formula, A is a C^*-algebra and $a_1, \ldots, a_n$ we can obtain the value $\phi^A(a_1, \ldots, a_n) \in \mathbb{R}$ by substituting a_i for x_i and using operations and norm of A.
- A formula without free-variables is a sentence. Given a sentence σ we write $A \models \sigma$ to mean $\sigma^A = 0$.

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$Th(A)$ the set of sentences that are "true" in A.

$$Th(A) = \{\sigma \mid A \vDash \sigma\} = \{\sigma \mid \sigma^A = 0\}$$

A and B are elementary equivalent ($A \equiv B$) if $Th(A) = Th(B)$.

Theorem (Keisler-Shelah)

- *$A \equiv A^{\mathcal{U}}$ for every ultrafilter $\mathcal{U}$*
- *$A \equiv B$ if and only if there is an ultrafilter $\mathcal{U}$ such that $A^{\mathcal{U}} \cong B^{\mathcal{U}}$. If the Continuum Hypothesis holds $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$, otherwise a larger index set may be needed.*

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Example

$$\Big(\sup_{\|x\|\leq 1}\sup_{\|y\|\leq 1}\|xy-yx\|\Big)^A=0$$

if and only if A is abelian.

$$\Big(\inf_{\|x\|=1}\|x^2\|\Big)^A=0$$

if and only if A is not abelian.

We focus on abelian C^*-algebras. Then $A\cong C(X)$. The following topological properties can be detected by the theory:

- being connected or disconnected
- having Lebesgue covering dimension $=$, $\leq$ or $\geq$ than n
- being indecomposable, hereditary indecomposable
- ..and much more!

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We focus on the 0-dimensional case. Every X is compact Hausdorff. A topological space is 0-dimensional if every open set contains a nontrivial clopen (closed+open). In this case the set of clopens $CL(X)$ is a Boolean algebra. It makes sense to talk about $CL(X) \equiv CL(Y)$ as Boolean algebras, in classical discrete model theory

Theorem (Eagle-V.)

$CL(X) \equiv CL(Y)$ if and only if $C(X) \equiv C(Y)$.

Theorem (??)

There are only $\aleph_0$-many theories of Boolean algebras.

Corollary

There are $\aleph_0$-many theories of abelian C^*-algebras of dimension 0. Moreover

- $C(\alpha+1) \equiv C(\beta+1) \equiv C(\beta\mathbb{N})$ *for every ordinal* α, β.
- $C(2^{\mathbb{N}}) \equiv C(\beta\mathbb{N} \setminus \mathbb{N})$

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We focus on the 0-dimensional case. Every X is compact Hausdorff. A topological space is 0-dimensional if every open set contains a nontrivial clopen (closed+open). In this case the set of clopens $CL(X)$ is a Boolean algebra. It makes sense to talk about $CL(X) \equiv CL(Y)$ as Boolean algebras, in classical discrete model theory

Theorem (Eagle-V.)

$CL(X) \equiv CL(Y)$ if and only if $C(X) \equiv C(Y)$.

Theorem (??)

There are only $\aleph_0$-many theories of Boolean algebras.

Corollary

There are $\aleph_0$-many theories of abelian C^*-algebras of dimension 0. Moreover

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Unfortunately

Theorem

- *For every $n > 0$, there are continuum many theories for n-dimensional abelian C^*-algebras.*
- *If $n > 0$, there is not a class of objects (groups, rings, lattices, etc) that can be associated "canonically" to abelian C^*-algebra preserving elementary equivalency.*

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Definition

- Let A be a C^*-algebra. Let Φ be a set of formulas with parameters from A. A **Φ-type** is a set of conditions of the form $\phi(\overline{x}) \in K$ where $\phi \in \Phi$ and $K \subseteq \mathbb{R}$ is compact.
- A Φ-type Σ is **consistent** if for every $\epsilon > 0$ and a finite $\Delta \subseteq \Sigma$ there is $\overline{a} \in A$ such that $d(\phi(\overline{a}), K) < \epsilon$ for every $\phi \in \Delta$.
- A Φ-type is **realized** if there is $\overline{a} \in A$ such that $\phi(\overline{a}) \in K$ for every $\phi \in \Sigma$,
- A is said **countably Φ-saturated** if every consistent countable Φ-type is realized.

Three degrees of saturation are important: 1-degree (Φ is the set of formulas of the form $\|P\|$, with P a *-polynomial of degree one), quantifier free (Φ is the set of quantifier free formulas), and full.

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If $C(X)$ is countably 1-degree saturated:

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and then a Theorem connecting the notions of saturation:

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If, in addiction, X does not have isolated point the following are equivalent

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in particular the theory of $C(2^{\mathbb{N}})$ has quantifier elimination.

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Are there other C^*-algebras with quantifier elimination, other than the obvious ones ($\mathbb{C}$, $\mathbb{C}^2$, $C(2^{\mathbb{N}})$, $M_2(\mathbb{C})$)?

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