Stochastic PDEs: an introduction

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Introduction

- What are SPDEs?
- A classical example
- How do we solve SPDEs?

An example arising in neurobiology

- A neuronal model
- The stochastic heat equation

PhD project

- The problem
- The questions

What are SPDEs?

SPDE stands for **S**tochastic **P**artial **D**ifferential **E**quation.

$$\boxed{\text{PDE "+ noise"} \longrightarrow \text{Stochastic PDE}}$$

- The noise is any random function which evolves in time: stochastic process

Tools

From analysis and probability theory.

Stochastic processes

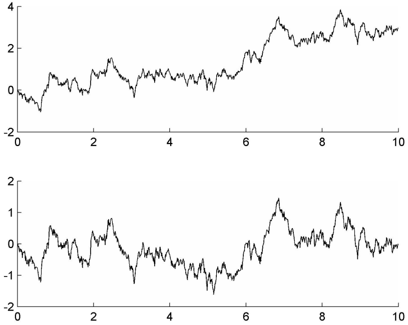


Figure: One realization of a Brownian Motion and a Brownian bridge.

PDE "+ noise"

Why do we study SPDEs?

- PDEs can describe several phenomena (physics, biology, medicine): heat or sound propagation, fluid flow, transport of substances, population dynamics, neuronal activity, traffic modelling, ...
- difficult to take into account every aspect of the problem..
- ..therefore useful to consider stochastic perturbations: stochastic heat equation, stochastic Navier-Stokes equation, stochastic Rall's linear cable equation, stochastic transport equations, forward rate dynamics, ...

A classical example: the vibrating string

- Consider a guitar string left outdoor during a sandstorm.
- The grains of sands hit the string continuously but irregularly.
- The number of grains hitting dx is essentially independent of the number hitting dy in time dt.
- $\dot{W}(dt, dx)$ is the random measure of the number of hits in (dt, dx), called space-time white noise

[Walsh, 1984] An Introduction to Stochastic Partial Differential Equations

- the position of the string $V(t,x)$ will satisfy a inhomogeneous wave equation driven by a space-time white noise.

$$\frac{\partial^2 V}{\partial t^2}(t,x) = \frac{\partial^2 V}{\partial x^2}(t,x) + \dot{W}$$

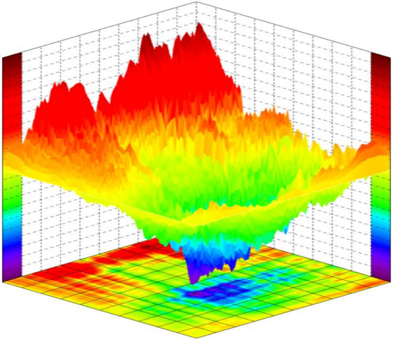
for $t > 0, x \in [a,b]$

- initial conditions

$$V(0,x) = \frac{\partial V}{\partial t}(t,x) = 0$$

for $x \in [a,b]$

Figure: One realization of the Brownian sheet W



The space-time white noise $\dot{W}$ is the "derivative" (in the distributional sense) of a Brownian sheet.

Which noise do we consider?

- of Brownian type: martingale processes, Itô calculus
- of fractional Brownian type: Gaussian process, stationary increments but not independent, non-Markovian, Hölder continuous paths
- of Lévy type: independent and stationary increments, jumps are allowed
- ...

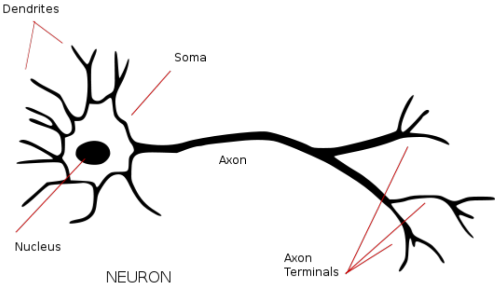
How do we solve SPDEs?

- define rigorously the equation: what is the noise?
- define the notion of solution: distribution? function? weak? mild?
- define the notion of integration with respect to the noise
- find the solution and study its regularity

- [Walsh, 1986] Brownian sheet approach
- [Da Prato, Zabczyk, 1992] Hilbert space description
- [Russo Vallois, 1993], [Lyons, 1994], [Zähle, 2002] Stieltjest-type integral, rought paths, pathwise method
- [Holden et al., 1996] Hida distributions, Wick product
- ...

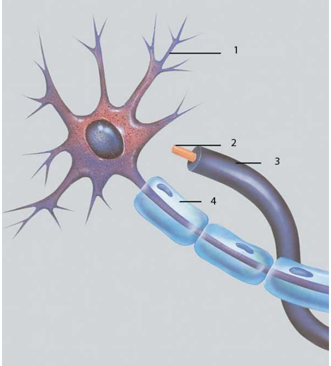
A neuronal model

- Neuronal cells comprise three main parts (the soma, the axon and the dendrites)
- with the junctions (called synapses) neurons emit and receive impulses of current.



Behaviour of a dendritic tree

A long thin cylinder which acts as an electrical cable.



The model

- $V(t,x)$ indicates the electric potential at time t and point x
- the potential is governed by a system of non-linear PDEs (Hodgkin-Huxley equations)
- well approximated by the cable equation

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} - V$$

- $F(t,x)$ indicates the current impulse arriving at time t in x

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} - V + F$$

The impulse F

- immense number of very small impulses
- take F stochastic
- F can be modelled by a white noise $\dot{W}$

The stochastic PDE

$$\begin{cases} \frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} - V + \dot{W}, & t > 0, x \in [0,\pi] \\ \frac{\partial V}{\partial x}(t,0) = \frac{\partial V}{\partial x}(t,\pi) = 0, & t > 0 \\ V(0,x) = 0, & x \in [0,\pi] \end{cases}$$

The deterministic heat equation

- the homogeneous case

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x)$$

- with initial condition $u(0,x) = \delta(x)$ has solution

$$u(t,x) = \Phi(t,x) := \frac{1}{\sqrt{4\pi t}} \exp\left\{-\frac{x^2}{4t}\right\}$$

which is called foundamental solution.

- the inhomogeneous case

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + g(t,x)$$

- with initial condition $u(0,x) = 0$ has solution

$$u(t,x) = \int_0^t \int_{-\infty}^{\infty} \Phi(t-s, x-y) g(s,y) dyds$$

that is the convolution between Φ and g.

The stochastic heat equation

$$\frac{\partial}{\partial t}V(t,x) = \frac{\partial^2}{\partial x^2}V(t,x) + \overbrace{\dot{W}(t,x)}^{\text{noise}}$$

- $\dot{W}(t,x) = \dot{W}(t,x,\omega)$ stochastic process indexed by (t,x)
- solution $V(t,x) = V(t,x,\omega)$ is a stochastic process

The solution

Formally replace $g(t,x)$ by $\dot{W}(t,x)$ to get

$$\begin{aligned} V(t,x) &= \int_0^t \int_{-\infty}^{\infty} \Phi(t-s,x-y) g(s,y) ds dy \\ &= \int_0^t \int_{-\infty}^{\infty} \Phi(t-s,x-y) \dot{W}(s,y) ds dy \\ &= \int_0^t \int_{-\infty}^{\infty} \Phi(t-s,x-y) W(ds,dy) \end{aligned}$$

We have a stochastic integral!

A peculiarity for the stochastic case

Space dimension 1: $t \in [0, T]$ and $x \in \mathbb{R}$.
The solution $V(t, x)$ is a proper function.

Space dimension 2: $t \in [0, T]$ and $x \in \mathbb{R}^2$.
The solution does not converge.

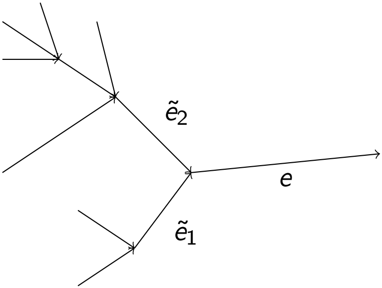
Higher dimensions: $t \in [0, T]$ and $x \in \mathbb{R}^n$.
The solutions do not converge.

In higher dimensions $n \geq 2$ solutions must be interpreted in the distributional sense.

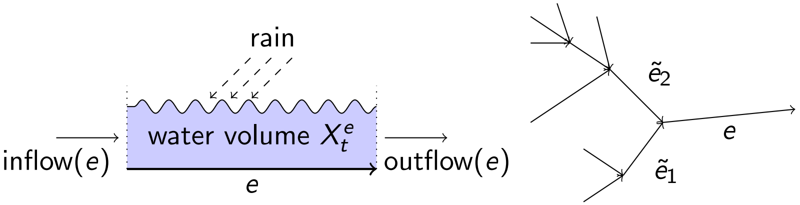
A PhD Project in Leeds (E. Issoglio, J. Voß)

An Abstract Model for Surface Water Flow: Stochastic Differential Equations on Graphs and related Fractal Limits

- simplified model for the flow of water on land
- continuous rainfall on land channeled into river-system: graph



- variable X_t^e: volume of water in edge e at time t



- conservation of water volume:

$$\frac{dX_t^e}{dt} = \sum_{\tilde{e} \to e} \text{outflow}(\tilde{e}) - \text{outflow}(e) + \text{rain}$$

- Assumption: the outflow of an edge e is proportional to the volume of water "contained" in the edge, *i.e.*

$$\text{outflow}(e) = \frac{1}{C(e)} X_t^e .$$

- The constant $C(e)$ depends on the length, depth and slope of the river modelled by the edge e
- get a system of SDEs for $(X_t^e)_{t\geq 0}^{e\in E}$

$$dX_t^e = \sum_{\tilde{e}\to e} \frac{1}{C(\tilde{e})} X_t^{\tilde{e}}\, dt - \frac{1}{C(e)} X_t^e\, dt + dR_t^e \qquad (*)$$

- R_t^e random total amount of rain feeding into edge e

Research directions

1. Study the system (*) both numerically and analytically, including conservation properties of the system.
2. Increase the number of edges in the graph and obtain refined models. Study the behaviour of the system of SDEs in the limit.
3. Consider the limiting object, where the "river system" now is a fractal, covering all of the two-dimensional land surface, and the system of SDEs is transformed into a (degenerate) Stochastic Partial Differential Equation. Study this system.
4. ...

Contact me for more info at `E.Issoglio@Leeds.ac.uk`

Conclusions:

- whenever we have a PDE where a stochastic component appears, we are dealing with a stochastic PDE
- SPDEs have a lot of applications in physics, biology, engineering, etc.
- to solve an SPDE one needs to give a meaning to the noise (e.g. space-time white noise)
- the techniques used in PDE theory are mimed for the SPDEs

- Available PhD Project at Leeds University: *An Abstract Model for Surface Water Flow*

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Thank you.