Linear relations in families of powers of elliptic curves

joint work with Laura Capuano

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Elliptic Curves

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$$F(X, Y) = 0,$$

for $F(X, Y) \in K[X, Y]$. If F has the special form

$$F = Y^2 - f(X),$$

where f is a degree 3 polynomial with simple roots, we call the curve an *Elliptic Curve*.

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$$\{(x,y) \in K^2 : F(x,y) = 0\} \cup \{O\}.$$

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Mordell (1922): The group of rational points of an elliptic curve over $\mathbb{Q}$ is finitely generated. Example:

$$Y^2 = X^3 - 82X,$$

the group of rational points is isomorphic to

$$\mathbb{Z}^3 \times \mathbb{Z}/2\mathbb{Z}.$$

Legendre family

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We can view it in two ways:

- as an elliptic curve over $\mathbb{Q}(\lambda)$, where λ is some variable;
- as a family of elliptic curves for $\lambda \in \mathbb{C} \setminus \{0,1\}$.

Consider the points

$$P_1(\lambda) = \left(2, \sqrt{2(2-\lambda)}\right) \quad \text{and} \quad P_2(\lambda) = \left(3, \sqrt{6(3-\lambda)}\right),$$

in $E_\lambda\left(\overline{\mathbb{Q}(\lambda)}\right)$.

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We have that this is a set of algebraic numbers and by Silverman Specialization Theorem, it is a set of bounded height.

This means that for every $d \in \mathbb{N}$

$$R \cap \left\{\alpha \in \overline{\mathbb{Q}} : [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\right\}$$

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Question

What about the λ_0 such that there are linearly independent $a, b \in \mathbb{Z}^2$ with $a_1 P_1(\lambda_0) = a_2 P_2(\lambda_0)$ and $b_1 P_1(\lambda_0) = b_2 P_2(\lambda_0)$, i.e., the points have finite order?

Theorem (Masser, Zannier)

There are at most finitely many λ_0 such that $P_1(\lambda_0)$ and $P_2(\lambda_0)$ are simultaneously torsion on E_{λ_0}.

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More generally, Masser and Zannier proved the Theorem for any complex distinct abscissas ($\neq 0, 1$).

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We are intersecting a curve with a countable union of curves in a space of dimension 3: Unlikely Intersections!

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$$\left\{\lambda_0 \in \mathbb{C} : \exists a \in \mathbb{Z}^3 \setminus \{0\} : a_1 P_1(\lambda_0) + a_2 P_2(\lambda_0) + a_3 P_3(\lambda_0) = O\right\}$$

is an infinite set of bounded height.

Theorem

There are at most finitely many λ_0 such that $P_1(\lambda_0)$, $P_2(\lambda_0)$ and $P_3(\lambda_0)$ satisfy two independent relations on E_{λ_0}.

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We were able to substitute 2, 3, 5 with pairwise distinct algebraic abscissas, and consider arbitrary many points

Theorem (B., Capuano)

Let $\mathcal{C} \subseteq \mathbb{A}^{2n+1}$ be an irreducible curve defined over $\overline{\mathbb{Q}}$ with coordinate functions $(x_1, y_1, \dots, x_n, y_n, \lambda)$, λ non-constant, such that, for every $j = 1, \dots, n$, the points $P_j = (x_j, y_j)$ lie on E_λ and there are no integers $a_1, \dots, a_n \in \mathbb{Z}$, not all zero, such that

$$a_1 P_1 + \cdots + a_n P_n = O,$$

identically on $\mathcal{C}$. Then there are at most finitely many $\underline{c} \in \mathcal{C}$ such that the points $P_1(\underline{c}), \dots, P_n(\underline{c})$ satisfy two independent relations on $E_{\lambda(\underline{c})}$.

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Fixing two independent relations gives something of dimension $n - 1$.

We are intersecting a curve with a countable union of $n - 1$-folds in a space of dimension $n + 1$: Unlikely Intersections!

The Zilber-Pink Conjecture

Conjecture

Let $\mathcal{A}$ be an abelian scheme over a variety defined over $\mathbb{C}$, and denote by $\mathcal{A}^{[c]}$ the union of its abelian subschemes of codimension at least c. Let $\mathcal{V}$ be an irreducible closed subvariety of $\mathcal{A}$. Then $\mathcal{V} \cap \mathcal{A}^{[1+\dim \mathcal{V}]}$ is contained in a finite union of abelian subschemes of $\mathcal{A}$ of positive codimension.

Theorem

Let $\mathcal{A}$ be an abelian scheme over a variety defined over $\overline{\mathbb{Q}}$ and suppose that $\mathcal{A}$ is isogenous to the fiber product of n isogenous elliptic schemes. Let $\mathcal{V}$ be an irreducible closed curve in $\mathcal{A}$. Then $\mathcal{V} \cap \mathcal{A}^{[2]}$ is contained in a finite union of abelian subschemes of $\mathcal{A}$ of positive codimension.