Sheaf Cohomology on Binoid Schemes

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Welcome Home Workshop 2014

Torino, 22 Dicembre 2014

UNIVERSITÄT OSNABRÜCK

DFG
GK 1916

Definition and Motivation

Monoid $M = (M, +, 0)$

Monoid Algebra $R[M]$

$$\begin{aligned} M &\longrightarrow R[M] \\ 0 &\longmapsto 1 \\ a &\longmapsto T^a \end{aligned}$$

and

$$T^a * T^b := T^{a+b}$$

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We can study Algebras from the combinatorics of Monoids

Toric Geometry, Tropical Geometry, Mirror Symetry, ...

Cannot represent 0-divisors

Cannot quotient out ideals of monoids

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Monoid $M = (M, +, 0)$

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Binoid $B = (B, +, 0, \infty)$

Binoid Algebra $R[B]$

If $N = (B, +, 0)$ then

$$R[B] := R[N] / T^{\infty}$$

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and we can try to describe combinatorially the algebraic properties of more algebras.

What algebras?

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Theorem (Eisenbud, Sturmfels 1996)**.** All and only the varieties closed under component-wise multiplication are binoidal varieties.

Integral Binoids

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$\mathbb{K}[B] = \mathbb{K}[x,y]/\langle xy\rangle$ non integral

$R = \mathbb{K}[x,y]/\langle x(x-y)\rangle$ comes from

$B = (X, Y \mid 2X = X + Y)$ integral but **non** cancellative

$X \neq Y$

M-sets

M monoid, S set

$$\begin{aligned} M \curvearrowright S : M \times S &\longrightarrow S \\ (a, s) &\longmapsto a + s \\ (0, s) &\longmapsto s \end{aligned}$$

$$(a+b)+s = a+(b+s)$$

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$$(a + b) + s = a + (b + s)$$

M binoid, (S, p) pointed set

$$M \curvearrowright S : M \times S \longrightarrow S$$
$$(a, s) \longmapsto a + s$$
$$(0, s) \longmapsto s$$
$$(\infty, s) \longmapsto p$$
$$(a, p) \longmapsto p$$

Simone Böttger, Holger Brenner

Introduction of Binoids and theoric bases

Bayarjargal Batsukh, Holger Brenner

Hilbert-Kunz multiplicity for M-sets

Ideals

M binoid, $I \subseteq M$, I an M-set

I ideal of M.

- $\infty \in I$
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Set to ∞ everything in I and leave everything else untouched.

$$M \longrightarrow M/_I$$
$$b \longrightarrow \begin{cases} \infty \text{ if } b \in I \\ b \text{ else} \end{cases}$$

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$\mathfrak{p} \subset M$ ideal is **prime** if $a + b \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$
Equiv. $M \setminus \mathfrak{p}$ is a monoid

$$\mathrm{Spec}(M) := \{\mathfrak{p} \subset M \mid \mathfrak{p} \text{ prime ideal}\}$$

Build Spec up

Theorem (Böttger 2014)**.** If M is a binoid with generating subset $G \subseteq M$ then every prime ideal of M is of the form $\langle A \rangle$ for some subset $A \subseteq G$.

$$(x, y, z \mid x + y = 2z)$$

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$\langle x \rangle, \langle y \rangle, \langle z \rangle$

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$\langle x, y, z \rangle$

Proposition (A. 2014)**.** $S \subseteq G$. $I = \langle S \rangle$ is prime iff for every relation r_i between generators of M, the elements of $S \cup \{\infty\}$ are either on both sides of r_i or on none.

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Algorithm.

IN: Generators and relations between them
OUT: List of prime ideals of M

Topology on Spec

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Localization: $M \setminus \mathfrak{p}$ monoid $\Rightarrow$ we can invert elements

$$M_{\mathfrak{p}} := {}^{-(M \setminus \mathfrak{p}) + M}\big/_{\sim_{\mathrm{loc}}}$$

$$= {}^{\{a - m \mid a \in M, m \in M \setminus \mathfrak{p}\}}\big/_{\sim_{\mathrm{loc}}}$$

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$f \in M$ **nilpotent** if
$\exists n \in \mathbb{N}$ s.t. $nf = \infty$

f **non** nilpotent

$$M_f := \{a - nf \mid a \in M, n \in \mathbb{N}\}$$

Structural Sheaf

Presheaf defined on the basis of fundamental open subsets

$$\begin{aligned} \mathrm{Top}(\operatorname{Spec} M) &\longrightarrow \mathrm{Bin} \\ D(f) &\longmapsto M_f \end{aligned}$$

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$(\mathrm{Spec}(M), \mathcal{O}_{\mathrm{Spec}\, M})$ is an **affine binoid scheme**.

If X is a topological space, $\mathcal{O}_X$ is a sheaf of binoids on X and $(X, \mathcal{O}_X)$ is locally isomorphic to affine binoid schemes, then $(X, \mathcal{O}_X)$ is a **binoid scheme**.

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The category of binoid schemes has finite products and coproducts.

TODO: Dive into Categorial properties

Simplicial complices

Let $V = \{1, \ldots, n\}$. An abstract simplicial complex is $\triangle \subseteq \mathcal{P}(V)$ closed under subsets, i.e.

$$F \in \triangle, G \subseteq F \Rightarrow G \in \triangle$$

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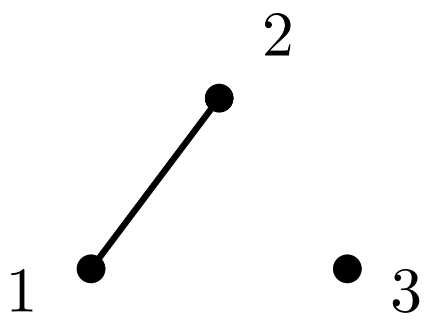
$V = \{1, 2, 3\}$

$\triangle = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}\}$



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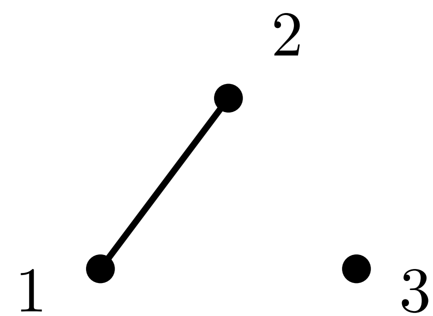

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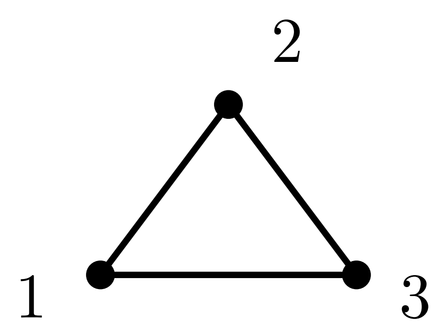

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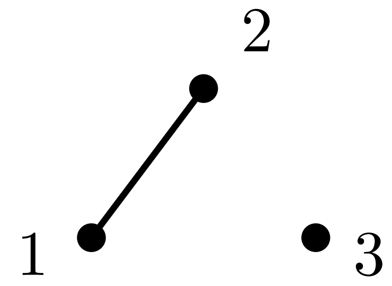

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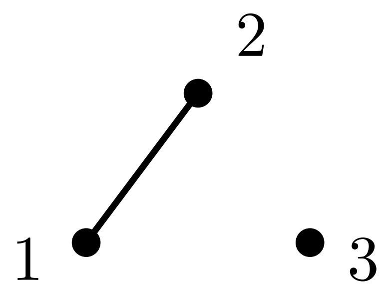

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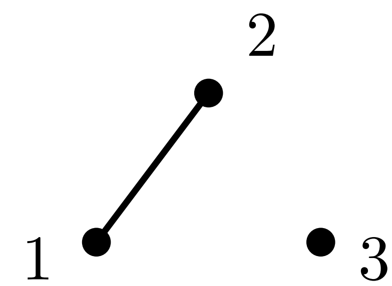

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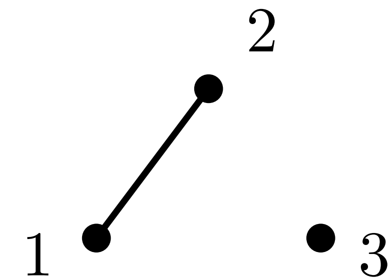

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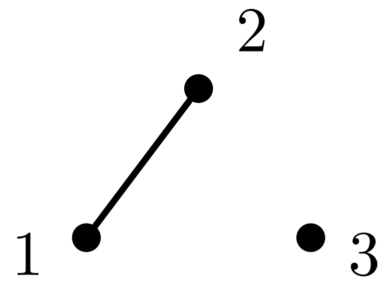

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$$\begin{aligned}\operatorname{Spec}(\triangle) &= \overline{\{\{3\}, \{1,2\}\}}_{\subset} \\ &= \{\{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}\end{aligned}$$

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$M_{\triangle_{x_i}}$ localization at a vertex

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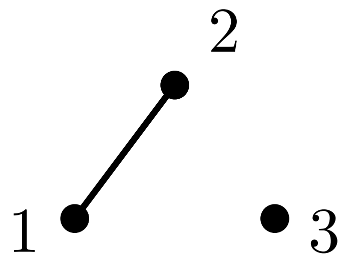

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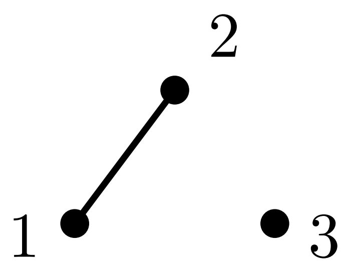

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Invert x_1 $\quad M_{\triangle_{x_1}} = (x_1, -x_1, \ldots \mid x_1 - x_1 = 0, x_3 = \infty) \cong (\mathbb{N} \times \mathbb{Z})^{\infty}$

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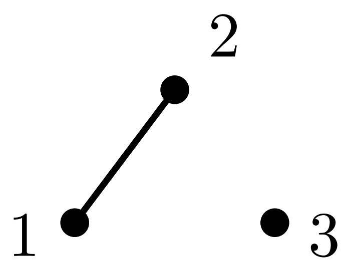

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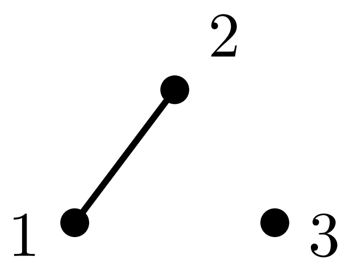

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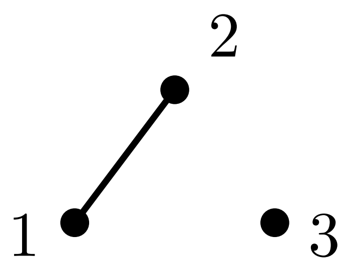

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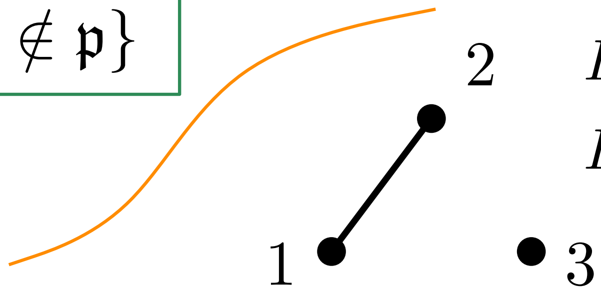
Invert x_1, x_3 $M_{\triangle_{x_1,x_3}} = (x_1, \ldots \mid 0 = \infty) = 0$

The sheaf of invertibles $\mathcal{O}^*$

$$D(x_i) := \{\mathfrak{p} \in \operatorname{Spec} \triangle \mid x_i \notin \mathfrak{p}\}$$



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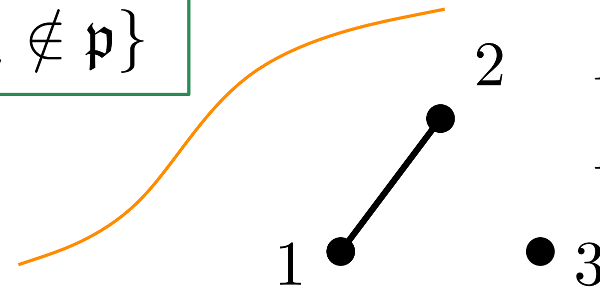

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$$D(x_i, x_j) = D(x_i) \cap D(x_j)$$

$$\begin{aligned} \mathcal{O}^* : \operatorname{Top}(\operatorname{Spec} M) &\longrightarrow \mathrm{Ab} \\ \mathcal{U} &\longrightarrow \mathcal{O}(\mathcal{U})^* \end{aligned}$$

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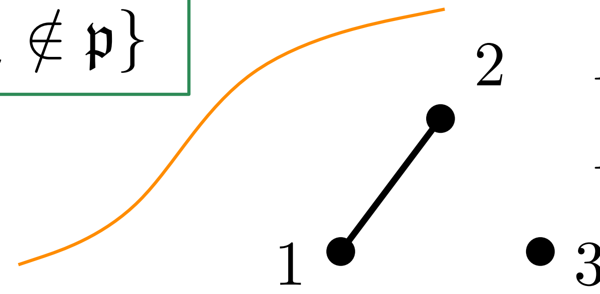

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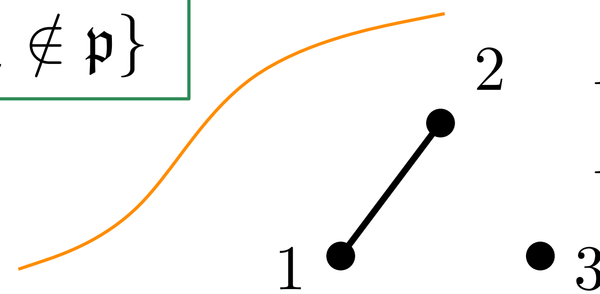

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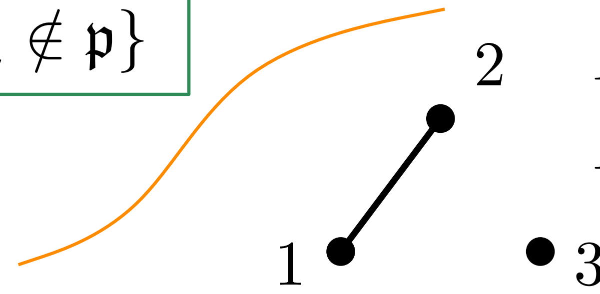

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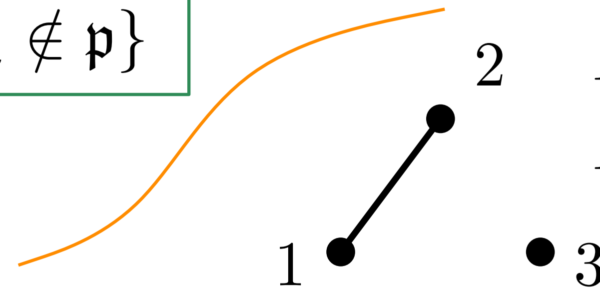

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Theorem (I. Pirashvili 2014)**.** For X of finite type, Čech Cohomology is Sheaf Cohomology

Vector Bundles

A vector bundle on a binoid scheme X is a sheaf $\mathcal{V}$ together with an action of $\mathcal{O}_X$ s.t. it is locally isomorphic to $\mathcal{O}_X^{\wedge n}$

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We call Čech-Picard complex the Čech complex of the sheaf of invertibles $\check{\mathcal{C}}^\bullet(X, \mathcal{O}^*)$

Puncturing

Since in the affine case everything vanishes, we puncture the spectrum by removing the (unique) maximal ideal $M_+ := M \smallsetminus M^*$

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Computed through the Čech-Picard complex on the covering given by the $D(x_i)$'s and intersections given by

$$D(x_{i_1}, \dots, x_{i_k}) = D(x_{i_1}) \cap \cdots \cap D(x_{i_k})$$

Local Čech-Picard complex

Our favourite (for now) example

$$\operatorname{Spec}^{\circ} \triangle = \{\{x_3\}, \{x_1, x_2\},$$
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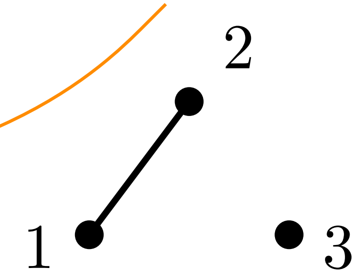

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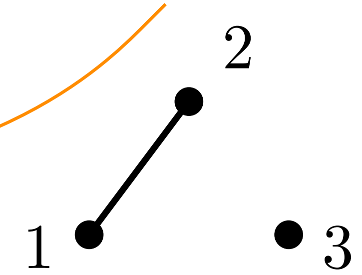

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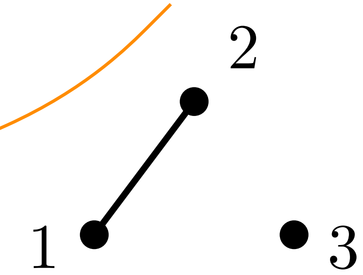

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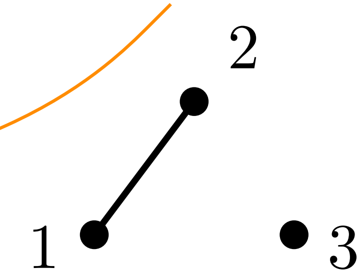

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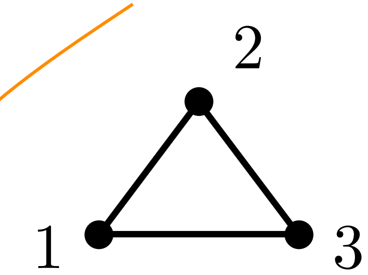

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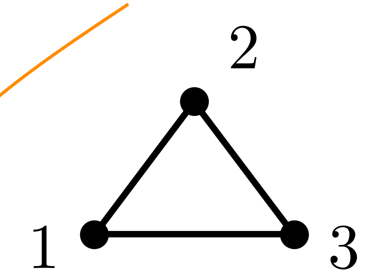

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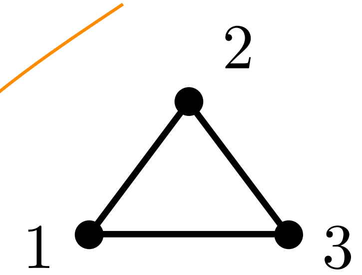

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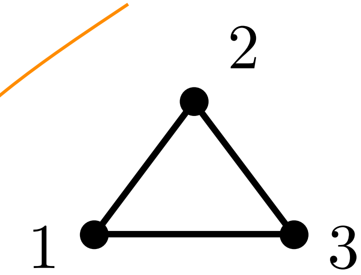

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They are the all and only such M-sets. Group with $\wedge_M$.

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Link complices and main result

Let $\triangle$ be a simplicial compex and $F \in \triangle$ one of its faces.

The link of F in $\triangle$ is
$\mathrm{lk}_{\triangle}(F) := \{G \in \triangle \mid F \cap X = \varnothing, F \cup G \in \triangle\}$

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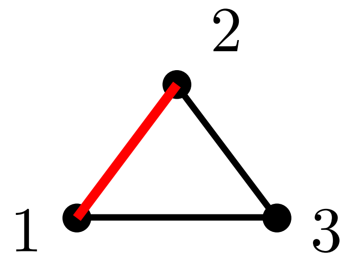

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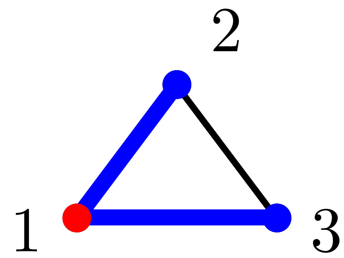

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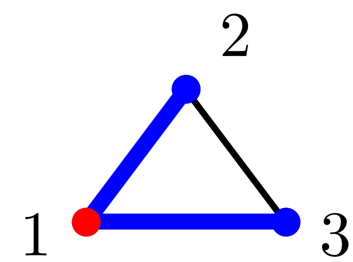

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Theorem (A. 2014). The cohomology of the local Čech-Picard complex of $\triangle$ can be computed with the following formulas

$$\mathrm{H}^0(\triangle, \mathcal{O}^*) = \mathbb{Z}^{\#\{0-\mathrm{dim}\ \text{facets of } \triangle\}}$$

$$\mathrm{H}^1(\triangle, \mathcal{O}^*) = \mathbb{Z}^{(\sum r_{v_i}) - \#\{0-\mathrm{dim}\ \text{non-facets of } \triangle\}}$$

$$\mathrm{H}^j(\triangle, \mathcal{O}^*) = \bigoplus_{v_i \in V} \mathrm{H}^{j-1}(\mathcal{C}^{\bullet}_{\mathrm{lk}_{\triangle}(v_i)})$$

- $\mathcal{C}^{\bullet}_{\mathrm{lk}_{\triangle}(v_i)} := \mathcal{C}^{\bullet}(\mathrm{lk}_{\triangle}(v_i), \mathbb{Z})$
- $r_{v_i} = \mathrm{rk}(\mathrm{H}^0(\mathcal{C}^{\bullet}_{\mathrm{lk}_{\triangle}(v_i)}))$
- $j \geq 2$

Conjectures and Corollaries

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OPEN

We can use these results in more general cases and/or to study the $\mathrm{Pic}^{\mathrm{loc}}$ of the Stanley-Reisner ring

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D. Eisenbud, J. Harris *The Geometry of Schemes*
D. Eisenbud, B. Sturmfels *Binomial Ideals*
J. Flores, C. Weibel *Picard groups and Class groups of Monoid Schemes*
R. Hartshorne *Algebraic Geometry*
A. Hatcher *Algebraic Topology*
J. López Peña, O. Lorscheid *Mapping* $\mathbb{F}_1$*-land: an overview of geometries over the field with one element*
E. Miller, B. Sturmfels *Combinatorial Commutative Algebra*
I. Pirashvili *On Cohomology and vector bundles over monoid schemes*

Thank You