Mitglied der Helmholtz-Gemeinschaft

JÜLICH FORSCHUNGSZENTRUM

The transfer function of the LIF model: A reduction from colored to white noise

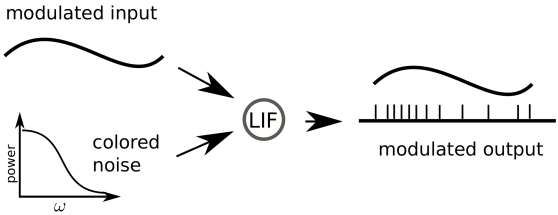
Jannis Schuecker
Institute of Neuroscience and Medicine (INM-6) and Institute for Advanced Simulation (IAS-6), Research Centre Jülich

Outline

- Motivation: Transfer function as a basic theoretical tool
- General reduction from colored to white noise
- Transfer function of the LIF neuron with **white** synaptic noise
- Transfer function of the LIF neuron with **colored** synaptic noise
- Comparison to direct simulations
- Outlook

Motivation

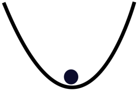
- Theory of correlated activity in recurrent networks relies on single neuron response to a modulation of its input, i.e. the transfer function $n(\omega)$
- Analytical expression for leaky integrate-and-fire (LIF) model only exists for white synaptic noise [Brunel et al. (1999)]
- For more realistic colored noise (synaptic filtering) only the limit for high frequencies is known [Brunel et al. (2001)]
- **Here: Analytical expression for colored noise transfer function valid for biological relevant frequencies (0-100Hz)**



General reduction from colored to white noise

- General first order system driven by white noise

$$\partial_s y = f(y,s) + \eta(s)$$



with unit variance white noise
$\langle \eta(t+s)\eta(t)\rangle = \delta(s)$

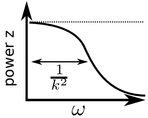
- Fokker-Planck equation for density $P(y,s)$

$$\partial_s P = -\partial_y \left(f(y,s) - \frac{1}{2}\partial_y \right) P$$

- General first order system driven by colored noise

$$\partial_s y = f(y,s) + \frac{z}{k}$$
$$k\partial_s z = -\frac{z}{k} + \eta(s)$$

colored noise



- small k

- Two-dimensional Fokker-Planck equation for density $P(y,z,s)$

$$k^2 \partial_s P = L_z P - k^2 \partial_y S_y P$$

with $L_z = \partial_z \left(\frac{1}{2}\partial_z + z\right)$ and flux operator $S_y = f(y,s) + \frac{z}{k}$

- Fokker-Planck equation $k^2\partial_s P = L_z P - k^2\partial_y S_y P$
- First order correction in k to z-marginalized probability flux?
- Solution using perturbative Ansatz in k

$$P(y,z,s) = P^{(0)}(y,s) + kP^{(1)}(y,z,s) + k^2P^{(2)}(y,z,s) + O(k^3)$$

- effective density

$$\tilde{P}(y,s) \equiv \int dz\, P = P^{(0)}(y,s) + kP_0^{(1)}(y,s) + O(k^2)$$

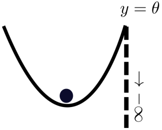
- effective flux

$$\nu_y(y,s) \equiv \int dz\, S_y P = \left(f(y,s) - \frac{1}{2}\partial_y \right) \tilde{P}(y,s) + O(k^2)$$

- effective Fokker-Planck equation (white-noise equation)

$$\partial_s \tilde{P}(y,s) = -\partial_y \nu_y(y,s) = -\partial_y \left(f(y,s) - \frac{1}{2}\partial_y \right) \tilde{P}(y,s)$$

Boundary conditions



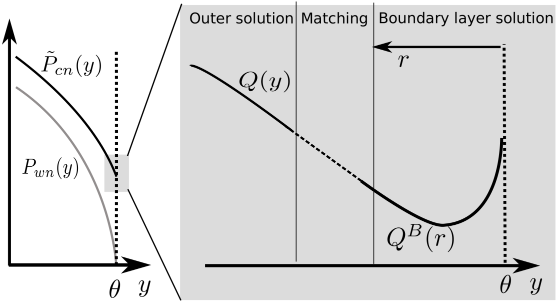
- Absorbing boundary at $y = \theta$
- Boundary condition given by matching of outer and boundary layer solution

$$\tilde{P}(\theta, s) = k\alpha\nu_y^{(0)}(s)$$

- Density value at shifted threshold $\tilde{\theta} = \theta + k\frac{\alpha}{2}$

$$\tilde{P}(\tilde{\theta}, s) = O(k^2)$$

- White-noise boundary condition



First conclusion

- Colored-noise problem

$$k^2\partial_s P = L_z P - k^2\partial_y S_y P$$

- White-noise problem with **adapted** boundary conditions

$$\partial_s P = -\partial_y \left(f(y,s) - \frac{1}{2}\partial_y \right) P$$

$$\tilde{\theta} = \theta + k\frac{\alpha}{2}$$

Transfer function of the LIF neuron with white synaptic noise

- LIF neuron model subject to white noise

$$\tau\dot{V} = -V + \mu(t) + \sigma\sqrt{\tau}\eta(t)$$

- Modulation of neuron's input

$$\mu(t) = \mu + \varepsilon\mu e^{i\omega t}$$

- Modulation of the neuron's response

$$\nu(t) = \nu(1 + n(\omega)e^{i\omega t})$$

where $n(\omega)$ is the **transfer function**

Transfer function of the LIF neuron with white synaptic noise

- Modulation causes perturbation of the Fokker-Planck equation

$$\partial_s \rho(x,s) = \underbrace{\partial_x (x + \partial_x)}_{\equiv \mathcal{L}_0} \rho(x,s) + e^{i\omega\tau s} \underbrace{(-G\,\partial_x)}_{\equiv \mathcal{L}_1} \rho(x,s).$$

- Perturbation ansatz $\rho(x,s) = \rho_0(x) + \rho_1(x)\, e^{i\omega\tau s}$ yields differential equation for the time modulated part of the density

$$i\omega\tau\,\rho_1 = \mathcal{L}_0 \rho_1 + \mathcal{L}_1 \rho_0.$$

- Ansatz $\rho(x,s) = u(x) q(x,s)$ with $u(x) = e^{-\frac{1}{4}x^2}$ yields

$$(i\omega\tau + a^\dagger a)\, q_1 = (G\, a^\dagger)\, q_0$$

with operators $a \equiv \frac{1}{2}x + \partial_x, a^\dagger \equiv \frac{1}{2}x - \partial_x$ and $a^\dagger a$ is Hermitian with $[a^\dagger, a] = 1$

Transfer function of the LIF neuron with white synaptic noise

- Perturbed Fokker Planck equation $(i\omega\tau + a^{\dagger}a)\, q_1 = (G\, a^{\dagger})\, q_0$
- Particular solution from eigenfunction $a^{\dagger}a\,(a^{\dagger}q_0) = a^{\dagger}q_0$

$$(i\omega\tau + \underbrace{a^{\dagger}a}_{=1})\; \underbrace{\frac{G}{1+i\omega\tau}\, a^{\dagger}q_0}_{q_p} = (G\, a^{\dagger})\, q_0$$

- Homogeneous solution as linear combination of parabolic cylinder functions $U(z,x), V(z,x)$
- Boundary conditions on flux

$$n(\omega) = \; u\,(S_0 q_1 + S_1 q_0)\big|_{x_\theta}$$

determine transfer function $n(\omega, \theta)$, in line with [Brunel et al. (1999)]

Transfer function of the LIF neuron with colored synaptic noise

- LIF neuron model with filtered synaptic noise in diffusion approximation

$$\tau \dot{V} = -V + I + \mu$$
$$\tau_s \dot{I} = -I + \sigma \sqrt{\tau} \eta(t)$$

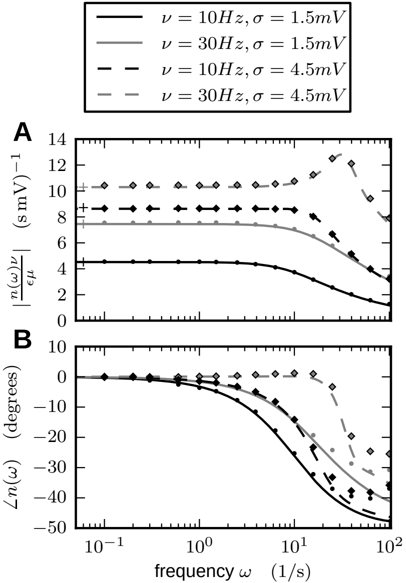
- Transfer function for colored noise obtained from white-noise solution with shifted threshold $\tilde{\theta} = \theta + k \frac{\alpha}{2}$ and shifted reset $\tilde{r} = r + k \frac{\alpha}{2}$

$$\begin{aligned} n_{cn}(\omega, \theta, r) &= n_{wn}(\omega, \tilde{\theta}, \tilde{r}) \\ &= \frac{G}{1 + i\omega\tau} \frac{\Phi'(\omega, x)\big|_{x_{\tilde{\theta}}}^{x_{\tilde{r}}}}{\Phi(\omega, x)\big|_{x_{\tilde{\theta}}}^{x_{\tilde{r}}}} \end{aligned}$$

with $\Phi(\omega, x) = u^{-1}(x)\, U(i\omega\tau - \frac{1}{2}, x)$

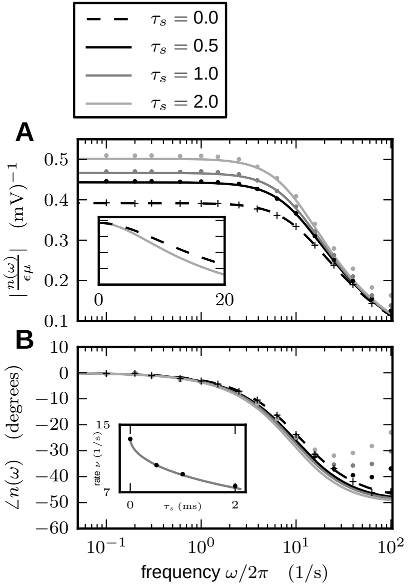
Results

- Simulation of transfer function with NEST
- For different working points the analytical prediction is in agreement with the simulations up to moderate frequencies ($100Hz$)
- Deviation for high frequencies expected since in the perturbative treatment we assume $\omega\tau k \ll 1$



Comparision to white noise

- Qualitative change compared to whote-noise case
- Synaptic filtering increases the dc-susceptibility counter to the behavior of the decreasing firing rate
- Synaptic filtering reduces the cutoff frequency (inset)



Shifted threshold and reset

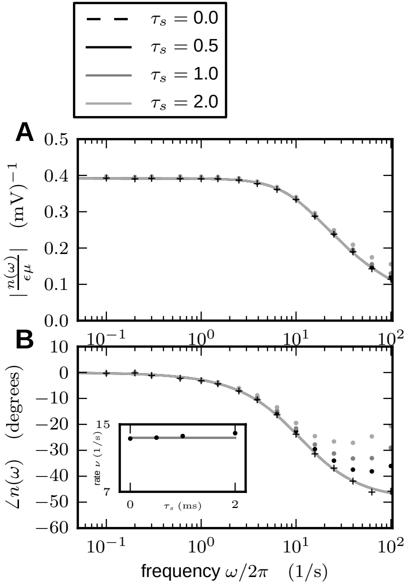
- Effective threshold
  $\tilde{\theta} = \theta + k\frac{\alpha}{2}, k = \sqrt{\frac{\tau_s}{\tau}}$
  valid for stationary and dynamic quantities
- Stationary firing rate

$$\nu_{cn} = \nu_{wn}(\tilde{\theta})$$

- Kepp firing rate constant for increasing τ_s by adapting threshold in the neuron:

$$\theta_c = \theta - k\frac{\alpha}{2}$$

- Dynamic transfer properties not altered for increasing τ_s



Summary and Outlook

- Colored-noise approximations for stationary but more importantly also for dynamic quantities are directly obtained by shifting the location of the boundaries in the white-noise solutions
- Include higher order terms to extent validity to higher frequencies
- Schuecker et al., arXiv:1411.0432 [cond-mat.stat-mech]

Collaborators:

Markus Diesmann

Moritz Helias