On 1D, 2D and 3D spline quasi-interpolation

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Spline spaces

- $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$
- $\Delta$ partition of $\Omega$
- The domain $\Omega$ is divided into a finite number of sub-domains $D_i$, $i = 1, \ldots, N$ by the partition $\Delta$
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Spline space

$$S^\mu_k(\Omega, \Delta) = \left\{ s \in C^\mu(\Omega) \mid s \mid_{D_i} \in \mathbb{P}_k(\mathbb{R}^d), \ i = 1, \ldots, N \right\}$$

$s \in S^\mu_k(\Omega, \Delta)$ is a piecewise polynomial of degree $k$ with $\mu$ order continuous (partial) derivatives in $\Omega$
A local spline quasi-interpolant (abbr. QI) of a function $f$ has the general form

$$Q : \mathcal{F} \rightarrow S^\mu_k(\Omega, \Delta)$$

$$Qf = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha(f) \phi_\alpha,$$
A local spline quasi-interpolant (abbr. QI) of a function \(f\) has the general form

\[
Q : \mathcal{F} \rightarrow S_k^\mu(\Omega, \Delta)
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where

- \(\{\phi_\alpha, \alpha \in A\}\) family of blending functions with compact support forming a partition of unity
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- $\{\lambda_\alpha, \alpha \in A\}$ family of local linear functionals defined on $\mathcal{F}$, expressed as linear combinations of values of $f$ at some points in a neighbourhood of $\text{supp} \phi_\alpha \cap \Omega$
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- \(Q\) exact on the space of polynomials of degree at most \(r\) \(\mathbb{P}_r(\mathbb{R}^d)\), i.e. \(Qp = p, \quad \forall p \in \mathbb{P}_r(\mathbb{R}^d), \, r \leq k\)
Spline quasi-interpolants on bounded domains

A local spline quasi-interpolant (abbr. QI) of a function $f$ has the general form

$$Q : \mathcal{F} \to S_{k}^{\mu}(\Omega, \Delta)$$

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- $\{\lambda_{\alpha}, \alpha \in A\}$ family of local linear functionals defined on $\mathcal{F}$, expressed as linear combinations of values of $f$ at some points in a neighbourhood of $\text{supp} \, \phi_{\alpha} \cap \Omega$
- $Q$ exact on the space of polynomials of degree at most $r$ $P_r(\mathbb{R}^d)$, i.e. $Qp = p$, $\forall p \in P_r(\mathbb{R}^d)$, $r \leq k$
- $\|f - Qf\|_{L^p(\Omega)} = O(h^{r+1})$, $1 \leq p \leq \infty$, $f$ sufficiently smooth function, $h$ maximum of the diameters of elements of $\Delta$
Outline

1D QIs in spline spaces of degree $k = 2, 3$ and smoothness $k - 1$

- Integral equations
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2D - QIs in spaces of quadratic splines on criss-cross triangulations
- Approximation of derivatives
- Error bounds for functions and derivatives
- Construction of NURBS surfaces
- Problems governed by PDEs
- QIs in spaces of $C^2$ cubic splines on uniform Powell-Sabin triangulations
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1D  QIs in spline spaces of degree $k = 2, 3$ and smoothness $k - 1$
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   - QIs in spaces of $C^2$ cubic splines on uniform Powell-Sabin triangulations

3D  QIs in spaces of $C^2$ quartic splines on uniform type-6 tetrahedral partitions
   - Reconstruction of volumetric data
   - Numerical integration
1D SPLINE SPACES
Univariate spline spaces of degree $k = 2, 3$ and smoothness $k - 1$, $S_{k}^{k-1}(\Omega, \Delta_n)$

- bounded interval $\Omega = [a, b]$
Univariate spline spaces of degree \( k = 2, 3 \) and smoothness \( k - 1 \), \( S_{k}^{k-1}(\Omega, \Delta_n) \)

- bounded interval \( \Omega = [a, b] \)
- uniform knot partition

\[
\Delta_n = \{x_{-k} = \ldots = x_{-1} = x_0 = a, \ x_i = a + ih, \ 1 \leq i \leq n - 1, \ b = x_n = x_{n+1} = \ldots = x_{n+k}\}.
\]
Univariate spline spaces of degree \( k = 2, 3 \) and smoothness \( k - 1 \), \( S_{k}^{k-1}(\Omega, \Delta_n) \)

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\[
\begin{array}{cccccccc}
 a & \ & \ & \ & \ & \ & b \\
 \hline
 x_0 & x_1 & \cdots & x_{n-1} & x_n \\
 x_{-1} & \cdots & \cdots & \cdots & \cdots \\
 \vdots & \cdots & \cdots & \cdots & \cdots \\
 x_{-k} & \cdots & \cdots & \cdots & \cdots \\
 b & \ & \ & \ & \ & \ & \end{array}
\]

- \( \left\{ B_j^k(x) \right\}_{j=0}^{n+k-1} \) basis of normalized B-splines defined on \( \Delta_n \), with \( \text{supp} \ B_j^k = [x_{j-k}, x_{j+1}] \), spanning the spline space \( S_{k}^{k-1}(\Omega, \Delta_n) \)
Spline quasi-interpolating projectors $P_k$ exact on $\mathbb{P}_k(\mathbb{R})$, $k = 2, 3$.
They are projectors, i.e. $P_k s = s$, $\forall s \in S_{k-1}^k(\Omega, \Delta_n)$.
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They are projectors, i.e. $P_k s = s$, $\forall s \in S_{k-1}^k(\Omega, \Delta_n)$

Quasi-interpolation nodes $\{t_i, i = 0, \ldots, 2n\}$, with

\[
\begin{align*}
t_{2i} &= x_i, & i = 0, \ldots, n \\
t_{2i-1} &= \frac{1}{2}(x_{i-1} + x_i), & i = 1, \ldots, n
\end{align*}
\]
Convergence properties

\( P_k \) are uniformly bounded independently of the uniform partition

\[ \|f - P_k f\|_\infty \leq C_k h^{k+1} \|f^{(k+1)}\|_\infty, \quad \text{with} \quad C_k = \begin{cases} \frac{7}{24}, & k = 2 \\ \frac{4}{9}, & k = 3 \end{cases} \]
Convergence properties

$P_k$ are uniformly bounded independently of the uniform partition

\[\text{Theorem}\]

For $f^{(k+1)}$ bounded, there holds

\[\|f - P_k f\|_\infty \leq C_k h^{k+1} \|f^{(k+1)}\|_\infty, \quad \text{with} \quad C_k = \begin{cases} \\frac{7}{24}, & k = 2 \\frac{4}{9}, & k = 3 \end{cases}\]

The operator $P_2$ is superconvergent at the quasi-interpolation nodes, i.e.

$(f - P_2 f)(t_i) = 0$, for $f \in \mathbb{P}_3(\mathbb{R})$,

$(f - P_2 f)(t_i) = O(h^4)$, for $f$ such that $\|f^{(4)}\|_\infty$ is bounded
Integral equations [Dagnino-Remogna-Sablonnière, 2014]

Linear Fredholm integral equations of the second kind

\[ \rho - \mathbf{T}\rho = \psi, \]

with

- \( \mathbf{T}\rho(x) = \int_{a}^{b} K(x, y)\rho(y)dy, \quad x \in \Omega \)
- \( \psi \in C(\Omega) \)
- \( K \in C(\Omega \times \Omega) \)
The four projection methods

1. **Galerkin method** → approximate equation
   \[ \rho_n^g - P_k TP_k \rho_n^g = P_k \psi \]

2. **Kantorovich method** → approximate equation
   \[ \rho_n^{ka} - P_k T \rho_n^{ka} = \psi \]

3. **Sloan’s iterated version** → approximate equation
   \[ \rho_n^s - TP_k \rho_n^s = \psi \]

4. **Kulkarni’s method** → approximate equation
   \[ \rho_n^{ku} - (P_k T + TP_k - P_k TP_k) \rho_n^{ku} = \psi \]
The approximate solutions

The approximate solution for each method is

(1) Galerkin method \( \rho^n_g = P_k \psi + \sum_{j=0}^{n+k-1} X_j B^k_j \)

(2) Kantorovich method \( \rho^n_{ka} = \psi + \sum_{j=0}^{n+k-1} X_j B^k_j \)

We have to solve a linear system and determine the unknowns \( \{X_j, j = 0, \ldots, n + k - 1\} \).

(3) Sloan’s iterated version: it is obtained as an iterate of Galerkin’s solution

\[
\rho^n_s = \psi + T \rho^n_g \quad \Rightarrow \quad \rho^n_s = \psi + \sum_{j=0}^{n+k-1} (\lambda_j(\psi) + X_j) TB^k_j,
\]

\( \{X_j, j = 0, \ldots, n + k - 1\} \) determined by Galerkin method.
The approximate solutions

Kulkarni’s method

\[ \rho_n^{ku} = \psi + \sum_{j=0}^{n+k-1} X_j B_j^k + \sum_{i=0}^{n+k-1} Y_i T B_i^k \]

The problem has \(2(n+k)\) unknowns \(\rightarrow\) linear system of \(2(n+k)\) equations

The system can be reduced to the solution of one system of \(n+k\) algebraic equations.

First we determine \(\{Y_i, \ j = 0, \ldots, n+k-1\}\) by solving the linear system, then we get \(\{X_j, \ j = 0, \ldots, n+k-1\}\)
Computation of the solutions

In order to construct the linear systems we have to evaluate different kinds of integrals. For example

\[ TB_j(x) = \int_a^b B_j(y) K(x, y) dy \rightarrow \text{suitable product quadrature formulas (PQF) with B-spline weight functions} \]

\[ \int_a^b K(t_j, y) \psi(y) dy \rightarrow \text{suitable Romberg’s quadrature formula} \]

\[ T\tilde{B}_i(t_j) = \int_a^b K(t_j, y) \tilde{B}_j(y) dy, \text{ with } \tilde{B}_i(x) = TB_i(x) \rightarrow \text{suitable Romberg’s quadrature formula} \]

\{t_j, j = 0, \ldots, 2n\} are the QI nodes.
Convergence orders of the solutions

**Theorem – case \( k = 2 \)**

Assume that the solution \( \rho \) has a bounded fourth derivative, then there holds

\[
\| \rho - \rho_n^g \|_\infty = O(h^3), \quad \| \rho - \rho_n^{ka} \|_\infty = O(h^3),
\]

\[
\| \rho - \rho_n^s \|_\infty = O(h^4), \quad \| \rho - \rho_n^{ku} \|_\infty = O(h^7)
\]

Superconvergence phenomenon at the set of QI nodes \( \{t_i, i = 0, \ldots, 2n\} \) in case of Galerkin, Kantorovich and Kulkarni methods

\[
\rho(t_i) - \rho_n^g(t_i) = O(h^4),
\]

\[
\rho(t_i) - \rho_n^{ka}(t_i) = O(h^4),
\]

\[
\rho(t_i) - \rho_n^{ku}(t_i) = O(h^8),
\]
Convergence orders of the solutions

**Theorem – case \( k = 3 \)**

Assume that the solution \( \rho \) has a bounded fourth derivative, then there holds

\[
\| \rho - \rho_n^g \|_\infty = O(h^4), \quad \| \rho - \rho_n^{ka} \|_\infty = O(h^4),
\]

\[
\| \rho - \rho_n^s \|_\infty = O(h^4 \varepsilon(h)), \quad \lim_{h \to 0} \varepsilon(h) = 0;
\]

\[
\| \rho - \rho_n^{ku} \|_\infty = O(h^8).
\]
Example

Integral equation \( \rho(x) - \int_{0}^{1} K(x, y) \rho(y) dy = \psi(x), \ x \in [0, 1], \) with

- **exact solution** \( \rho(x) = \exp(-x) \cos(x) \)
- **kernel** \( K(x, y) = \exp(xy) \)
- **function**
  \[
  \psi(x) = \exp(-x) \cos(x) + \frac{\exp(x-1)(\cos(1)(1-x)-\sin(1))+x-1}{x^2-2x+2}
  \]
Integral equation $\rho(x) - \int_0^1 K(x, y)\rho(y)dy = \psi(x), \ x \in [0, 1]$, with

- exact solution $\rho(x) = \exp(-x)\cos(x)$
- kernel $K(x, y) = \exp(xy)$
- function
  $$\psi(x) = \exp(-x)\cos(x) + \frac{\exp(x-1)(\cos(1)(1-x)-\sin(1)) + x-1}{x^2 - 2x + 2}$$

We compute the maximum absolute error for increasing values of $n$

$$e_n = \max_{z \in G} |\rho(z) - \rho_n(z)|,$$

where $G$ is a set of 1500 equally spaced points in $[0, 1]$. We also compute the numerical convergence order (NCO).
### Methods based on $P_2$

<table>
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<tr>
<th>$n$</th>
<th>$e_n^g$</th>
<th>$NCO_g$</th>
<th>$e_n^{ka}$</th>
<th>$NCO_{ka}$</th>
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### Methods based on $P_3$

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Example

We compute the maximum absolute error at the QI nodes \( \{t_j, j = 0, \ldots, 2n\} \) for the spline projector \( P_2 \)

\[
es_n = \max_{z \in \{t_j, j = 0, \ldots, 2n\}} |\rho(z) - \rho_n(z)|
\]

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Superconvergence
Work in progress

Spline quasi-interpolation for the solution of:

- integral equations with Green’s function kernels
- weakly singular integral equations
- line integral equations
2D SPLINE SPACES
Bivariate quadratic spline spaces on criss-cross triangulations $\Delta_{mn}: S^1_2(\Omega, \Delta_{mn})$

Let:

- $\Omega = [a, b] \times [c, d]$ be a rectangular domain
- $m, n$ be positive integers
- $\bar{x} = (x_i)_{i=-2}^{m+2}$, where $x_{-2} = x_{-1} = x_0 = a < x_1 < \ldots < x_{m-1} < b = x_m = x_{m+1} = x_{m+2}$
- $\bar{y} = (y_j)_{j=-2}^{n+2}$, where $y_{-2} = y_{-1} = y_0 = c < y_1 < \ldots < y_{n-1} < d = y_n = y_{n+1} = y_{n+2}$
- $\Delta_{mn}$ be the corresponding criss-cross triangulation
Bivariate quadratic spline spaces on criss-cross triangulations $\Delta_{mn}$: $S_2^1(\Omega, \Delta_{mn})$

Let:

- $\Omega = [a, b] \times [c, d]$ be a rectangular domain
- $m, n$ be positive integers
- $\bar{x} = (x_i)_{i=-2}^{m+2}$, where $x_{-2} = x_{-1} = x_0 = a < x_1 < \ldots < x_{m-1} < b = x_m = x_{m+1} = x_{m+2}$
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- $\Delta_{mn}$ be the corresponding criss-cross triangulation
Example: $S_2^1(\Omega, \Delta_{mn})$ [BOX SPLINES: de Boor, Chui, Dagnino, Dahmen, Höllig, Lyche, Micchelli, Sablonnière, Schumaker, Wang, ...]

- classical spanning functions
- data points also outside the domain
First method [Sablonnière, Dagnino, Demichelis, Lamberti, Remogna, ...]

- spanning functions with support completely in the domain
- data points inside or on the boundary of the domain
Second method [Remogna, 2010]

- classical spanning functions
- data points inside or on the boundary of the domain
Spanning set $\mathcal{B}$ [Sablonnière, 2003]

$$\mathcal{B} = \left\{ B_{ij} \right\}_{i=0, j=0}^{m+1, n+1}$$

collection of $(m + 2)(n + 2)$ quadratic B-splines, with support $\Sigma_{ij}$, spanning $S^1_2(\Omega, \Delta_{mn})$
Dimension and basis

\[ \text{dim } S_2^1(\Omega, \Delta_{mn}) = (m + 2)(n + 2) - 1 \]
Dimension and basis

\[ \dim S^1_2(\Omega, \Delta_{mn}) = (m + 2)(n + 2) - 1 \]

\[ \Downarrow \]

The B-splines in the spanning set \( B \) are linearly dependent
Dimension and basis

\[ \dim S^1_2(\Omega, \Delta_{mn}) = (m + 2)(n + 2) - 1 \]

\[ \Downarrow \]

The B-splines in the spanning set \( B \) are linearly dependent

In order to obtain a basis, we have to remove one B-spline from \( B \), with \( C^1 \) smoothness everywhere or with \( C^0 \) smoothness on the boundary of its support
Optimal spline quasi-interpolants

Optimal spline quasi-interpolants exact on $\mathbb{P}_2(\mathbb{R}^2)$

\[ Q : \mathcal{F} \rightarrow S_2^1(\Omega, \Delta_{mn}) \]

\[ f(x, y) \approx Qf(x, y) \]
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Optimal spline quasi-interpolants exact on $\mathbb{P}_2(\mathbb{R}^2)$

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\[ f(x, y) \approx Qf(x, y) \]

Quasi-interpolation nodes:
\[ \{M_{ij} = (s_i, t_j)\}, \quad s_i = \frac{x_{i-1} + x_i}{2}, \quad t_j = \frac{y_{j-1} + y_j}{2} \]
\[ \{A_{ij} = (x_i, y_j)\} \]
Approximation of partial derivatives [Dagnino-Remogna-Sablonnière, 2013]

\[ D^\alpha f(\bar{x}, \bar{y}) \approx D^\alpha Qf(\bar{x}, \bar{y}) \]

with:

- \( \alpha = (\alpha_1, \alpha_2), \, |\alpha| = \alpha_1 + \alpha_2, \, 0 \leq |\alpha| \leq 2 \)
- \( D^\alpha = \frac{\partial |\alpha|}{\partial^{\alpha_1} x \partial^{\alpha_2} y} \)
- \((\bar{x}, \bar{y}) \in T \) triangle in \( \Delta_{mn} \) for \(|\alpha| = 0, 1\)
- \((\bar{x}, \bar{y}) \in \text{int}(T), \) \( T \) triangle in \( \Delta_{mn} \) for \(|\alpha| = 2\)
Let $f \in C^\nu(\Omega)$, with $0 \leq |\alpha| \leq \nu \leq 2$, $|\alpha| = 0, 1, 2$, then

$$\| D^\alpha (f - Qf) \|_{\infty, T} \leq \overline{C}_{|\alpha|, \nu} \left( \frac{h_T}{h^*_T} \right)^{|\alpha|} h_T^{\nu - |\alpha|} \omega \left( D^\nu f, \frac{h_T}{2}, \Omega \right).$$

If, in addition, $f \in C^3(\Omega)$, then

$$\| D^\alpha (f - Qf) \|_{\infty, T} \leq \overline{C}_{|\alpha|, 3} \left( \frac{h_T}{h^*_T} \right)^{|\alpha|} h_T^{3 - |\alpha|} \| D^3 f \|_\Omega,$$

with $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$, $h_T = \max \{ h_i, k_j \}$, $h^*_T = \min \{ h_i, k_j \}$.

Global results only for $|\alpha| = 0, 1$. 

[Dagnino-Remogna-Sablonnière, 2013]
For $|\alpha| = 0$, the error bounds are independent of the mesh ratios $\left(\frac{h_T}{h_T^*}\right) \Rightarrow Qf \rightarrow f$ in $T$ as $h_T \rightarrow 0$. 

\[ \| D^\alpha (f - Qf) \|_\infty \] [Dagnino-Remogna-Sablonnière, 2013]
For $|\alpha| = 0$, the error bounds are independent of the mesh ratios $\left(\frac{h_T}{h_T^*}\right) \Rightarrow Qf \to f$ in $T$ as $h_T \to 0$.

For $|\alpha| = 1, 2$, the error bounds depend on the mesh ratios $\left(\frac{h_T}{h_T^*}\right)$. When such ratios are bounded $\Rightarrow D^\alpha Qf \to D^\alpha f$ in $T$ as $h_T \to 0$. 

Error $\|D^\alpha (f - Qf)\|_\infty$ [Dagnino-Remogna-Sablonnière, 2013]
For $|\alpha| = 0$, the error bounds are independent of the mesh ratios $\left(\frac{h_T}{h_T^*}\right) \Rightarrow Qf \rightarrow f$ in $T$ as $h_T \rightarrow 0$.

For $|\alpha| = 1, 2$, the error bounds depend on the mesh ratios $\left(\frac{h_T}{h_T^*}\right)$. When such ratios are bounded $\Rightarrow D^\alpha Qf \rightarrow D^\alpha f$ in $T$ as $h_T \rightarrow 0$.

Such ratios are bounded for example in case of uniform triangulation or $\gamma$-quasi uniform partitions ($0 < h/h^* \leq \gamma$, $\gamma > 1$ constant).
Example

- Test function on $\Omega = [-4, 4]^2$
  
  $$f(x, y) = 3(1 - x)^2 e^{(-x^2 - (y+1)^2)} - 10 \left( \frac{x}{5} - x^3 - y^5 \right) e^{(-x^2 - y^2)} - \frac{1}{3} e^{(-(x+1)^2 - y^2)}$$

- $G$ is a $300 \times 300$ uniform rectangular grid of evaluation points in $\Omega$

- $f_{\text{error}} = \max_{(u,v) \in G} |(f - Qf)(u, v)|$

- $D^{(\alpha_1, \alpha_2)} f_{\text{error}} = \max_{(u,v) \in G} |D^{(\alpha_1, \alpha_2)}(f - Qf)(u, v)|, |\alpha| = 1$
The graphs of $f$, $Qf$ and $|f - Qf|$ computed on the grid $G$, considering a uniform triangulation with $m = n = 128$.
Example

The graphs of $D^{(1,0)} f$, $D^{(1,0)} Qf$ and $|D^{(1,0)} (f - Qf)|$ computed on the grid $G$, considering a uniform triangulation with $m = n = 128$.
Example

The graphs of $D^{(0,1)} f$, $D^{(0,1)} Qf$ and $|D^{(0,1)}(f - Qf)|$ computed on the grid $G$, considering a uniform triangulation with $m = n = 128$
Quadratic spline space $S_2^{(\bar{\mu}^x, \bar{\mu}^y)}(\Omega, \Delta_{mn})$ [Dagnino-Lamberti-Remogna, 2012]

Let $S_2^{(\bar{\mu}^x, \bar{\mu}^y)}(\Omega, \Delta_{mn})$ be the space of bivariate quadratic piecewise polynomials on $\Delta_{mn}$, where

- $\bar{\mu}^x = (\mu^x_i)_{i=1}^{m-1}$ and $\bar{\mu}^y = (\mu^y_j)_{j=1}^{n-1}$ are vectors whose elements can be 1, 0, -1 and denote the smoothness $C^1$, $C^0$, $C^{-1}$, respectively, across the inner grid lines $x - x_i = 0, i = 1, \ldots, m - 1$ and $y - y_j = 0, j = 1, \ldots, n - 1$,
- while the smoothness across all oblique mesh segments\(^1\) is $C^1$

\(^1\text{We call mesh segments the line segments that form the boundary of each triangular cell of } \Delta_{mn}\).
Given \( s \in S_2^{(\bar{\mu}^x, \bar{\mu}^y)}(\Omega, \Delta_{mn}) \), we denote by

- \( L_x^0 \ (L_x^{-1}) \) the number of \underline{vertical} grid lines \( x - x_i = 0, \ i = 1, \ldots, m - 1 \) across which \( s \) has \( C^0 \ (C^{-1}) \) smoothness
- \( L_y^0 \ (L_y^{-1}) \) the number of \underline{horizontal} grid lines \( y - y_j = 0, \ j = 1, \ldots, n - 1 \) across which \( s \) has \( C^0 \ (C^{-1}) \) smoothness

\[ \downarrow \]

\[ \dim S_2^{(\bar{\mu}^x, \bar{\mu}^y)}(\Omega, \Delta_{mn}) = d_1 + d_2 + d_3, \]

where

\[ d_1 = (m + 2)(n + 2) - 1 \]
\[ d_2 = (n + 1)L_x^0 + (m + 1)L_y^0 \]
\[ d_3 = (2n + 3 + L_y^0 + L_y^{-1})L_x^{-1} + (2m + 3 + L_x^0 + L_x^{-1})L_y^{-1} + L_x^{-1}L_y^{-1} \]
Spanning set $\mathcal{B}$

\[ \mathcal{B} = \left\{ B_{ij} \right\}_{i=0,j=0}^{M-1,N-1} \]

collection of $M \cdot N$ quadratic B-splines with multiple knots, spanning $S_2^{(\bar{\mu}^x, \bar{\mu}^y)}(\Omega, \Delta_{mn})$, where

- $M = 2 + \sum_{i=1}^{m-1} m^x_i$, $N = 2 + \sum_{j=1}^{n-1} m^y_j$
- $\bar{m}^x = (m^x_i)_{i=1}^{m-1}$, $m^x_i = 2 - \mu^x_i$, $\bar{m}^y = (m^y_j)_{j=1}^{n-1}$, $m^y_j = 2 - \mu^y_j$

vectors of knot multiplicity

$B_{ij}$ smoothness and support, containing multiple knots, change as the number of triangular cells on which the function is nonzero is reduced
- A thick line corresponds to a double knot.
- A dotted line corresponds to a triple knot.
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B-spline basis

\[ \#B = M \cdot N > \dim S_2^{(\bar{\mu}^\xi, \bar{\mu}^\eta)}(T_{mn}) \]

i.e. the elements of $B$ are linearly dependent.
B-spline basis

\[ \#B = M \cdot N > \dim S_2^{(\bar{\mu}^\xi, \bar{\nu}^\eta)}(T_{mn}) \]

i.e. the elements of \( B \) are linearly dependent.

In order to obtain a basis, we have to remove one B-spline for each subdomain, with \( C^1 \) smoothness everywhere or with \( C^0 \) smoothness only on the boundary of its support.

![Diagram of subdomains](image)

- A thick line corresponds to a double knot
- A dotted line corresponds to a triple knot
Applications to NURBS surfaces

We provide:

- bidirectional net of control points \((P_{ij}), P_{ij} \in \mathbb{R}^3\)
- real positive weights \((w_{ij})\)
- suitable knot vectors \(\bar{x}\) and \(\bar{y}\) in the parametric domain \(\Omega\)
Applications to NURBS surfaces

We provide:

- bidirectional net of control points \( (P_{ij}, P_{ij} \in \mathbb{R}^3) \)
- real positive weights \( (w_{ij}) \)
- suitable knot vectors \( \bar{x} \) and \( \bar{y} \) in the parametric domain \( \Omega \)

and we define the NURBS surface

\[
S(x, y) = \frac{\sum_{ij} w_{ij} P_{ij} B_{ij}(x, y)}{\sum_{ij} w_{ij} B_{ij}(x, y)} = \sum_{ij} P_{ij} R_{ij}(x, y), \quad (x, y) \in \Omega
\]

with

\[
R_{ij}(x, y) = \frac{w_{ij} P_{ij} B_{ij}(x, y)}{\sum_{rs} w_{rs} B_{rs}(x, y)}
\]
The B-splines in $\mathcal{B}$ are non negative and satisfy the property of unity partition $\Rightarrow S(x, y)$ has convex hull and affine transformation invariance properties.

$\bar{x}$ and $\bar{y}$ have simple knots $\Rightarrow S(x, y)$ is $C^1$.

B-spline locality property $\Rightarrow S(x, y)$ interpolates the control points $P_{ij}$ whose pre-images are the corner points of each subdomain.

In case $w_{ij} = w$, $\forall (i, j)$, then $S(x, y)$ is a B-spline surface. If, in addition, we consider a functional parametrization, $S(x, y)$ is the spline function defined by the well known bivariate “variation diminishing” operator, reproducing bilinear functions.
We want to reconstruct a bar bell by a suitable quadratic NURBS surface
We consider the bidirectional net of control points \((P_{ij})_{i=0,j=0}^{8,12}\).
Example

We consider the bidirectional net of control points \((P_{ij})_{i=0,j=0}^{8,12}\),

and the weights \((w_{ij})_{i=0,j=0}^{8,12}\)

\[
\begin{align*}
    w_{00} &= w_{10} = w_{20} = w_{30} = w_{40} = w_{50} = w_{60} = w_{70} = w_{80} = 1, \\
    w_{0j} &= w_{2j} = w_{4j} = w_{6j} = w_{8j} = 1, \\
    w_{1j} &= w_{3j} = w_{5j} = w_{7j} = \frac{\sqrt{2}}{2}, j = 1, \ldots, 11, \\
    w_{0,12} &= w_{1,12} = w_{2,12} = w_{3,12} = w_{4,12} = w_{5,12} = w_{6,12} = w_{7,12} = w_{8,12} = 1.
\end{align*}
\]
Example

If we assume:

- \( \Omega = [0, 1] \times [0, 1] \)
- \( \bar{x} = (0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, 1, 1, 1) \), \( \bar{y} = (0, 0, 0, \frac{1}{7}, \frac{1}{7}, \frac{2}{7}, \frac{2}{7}, \frac{3}{7}, \frac{3}{7}, \frac{4}{7}, \frac{4}{7}, \frac{5}{7}, \frac{5}{7}, \frac{6}{7}, 1, 1, 1) \)
Example

If we assume:

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then we can model the object by the \( C^0 \) NURBS surface

\[
S(x, y) = \frac{\sum_{i=0}^{8} \sum_{j=0}^{12} w_{ij} P_{ij}(x, y)}{\sum_{i=0}^{8} \sum_{j=0}^{12} w_{ij} B_{ij}(x, y)}
\]
Applications to elliptic diffusion-type problems with mixed boundary conditions [Cravero-Dagnino-Remogna, 2012]

Let:
- \( \Omega \subset \mathbb{R}^2 \) be an open, bounded and Lipschitz domain
- \( \emptyset \subseteq \Gamma_D, \Gamma_N \subseteq \partial \Omega, \partial \Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N \) and \( \Gamma_D \cap \Gamma_N = \emptyset \)

\[
\begin{aligned}
- \nabla \cdot (K \nabla u) &= f, \quad \text{in } \Omega \\
\frac{\partial u}{\partial n_K} &= g_N, \quad \text{on } \Gamma_N, \quad \text{(Neumann conditions)} \\
u &= g, \quad \text{on } \Gamma_D \quad \text{(Dirichlet conditions)}
\end{aligned}
\]

where
- \( K \in \mathbb{R}^{2 \times 2} \) is a symmetric positive definite matrix
- \( n_K = Kn \) is the outward conormal vector on \( \Gamma_N \)
- \( f \in L^2(\Omega) \)
- \( g_N \in L^2(\Gamma_N) \)
- \( g \in H^{1/2}(\Gamma_D) \) (\( g \) is the trace on \( \Gamma_D \) of function in \( H^1(\Omega) = \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega), |\alpha| \leq 1\} \))
Reconstruction of the domain

$\Omega$ domain in $\mathbb{R}^2$, exactly described through the parametrization

$$G : \Omega_0 \rightarrow \bar{\Omega}, \quad G(\xi, \eta) = \begin{pmatrix} x \\ y \end{pmatrix}$$

expressed as quadratic NURBS surface

$$G(\xi, \eta) = \sum_{ij} P_{ij} R_{ij}(\xi, \eta),$$

with $\{P_{ij}\}, \, P_{ij} \in \mathbb{R}^2$, bidirectional net of control points
Numerical solution

Weak formulation

\[\Rightarrow\]

Galerkin method
(discretized problem depending on the parameter \( h > 0 \))
Numerical solution

Weak formulation

\[ u_h \approx u \cdot \sum_{ij} q_{ij} (R_{ij} \circ G^{-1}) \]

$q_{ij}$ to be determined by solving a linear system
Example

\[
\begin{aligned}
-\Delta u &= f, \quad \text{in } \Omega, \\
 u &= 0, \quad \text{on } \Gamma_D = \partial \Omega,
\end{aligned}
\]

exact solution
\[u(x, y) = \sin(\pi x) \sin(\pi y)\]

- Reproduce the domain → introduce a discontinuity in the first derivative and create the corners → two approaches:
  1. place two control points at the same location in physical space;
  2. use suitable double knots in the knot vectors.
Example

\[
\begin{aligned}
\begin{cases}
-\Delta u = f, & \text{in } \Omega, \\
u = 0, & \text{on } \Gamma_D = \partial \Omega,
\end{cases}
\end{aligned}
\]

exact solution
\[u(x, y) = \sin(\pi x) \sin(\pi y)\]

Reproduce the domain \(\rightarrow\) introduce a discontinuity in the first derivative and create the corners \(\rightarrow\) two approaches:

1. place two control points at the same location in physical space;
2. use suitable double knots in the knot vectors.

In the first case, we ensure that the basis has \(C^1\) continuity throughout the interior of the domain.
Example – Approach 1: Double control point

Reproduce the domain:
\( \bar{\xi} = (0, 0, 0, 0.5, 1, 1, 1), \bar{\eta} = (0, 0, 0, 1, 1, 1) \)

\[
G(\xi, \eta) = \sum_{i=0}^{3} \sum_{j=0}^{2} P_{ij} B_{ij}(\xi, \eta) \quad (\xi, \eta) \in \Omega_0
\]
Example – **Approach 1: Double control point**

- Perform $h$-refinement, considering $m = 2, 4, 8, 16, 32$, $n = 1, 2, 4, 8, 16$, and smoothness vectors $\bar{\mu}^\xi, \bar{\mu}^\eta$ with elements equal to one.
- To obtain a basis, we have to neglect one B-spline either with $C^1$ smoothness everywhere or with $C^0$ smoothness on the boundary of its support.

The graphs of

(a) the exact solution
(b) the approximation computed with $m = 8, n = 4$
(c) the discrete $L^\infty$-norm of the error computed on a $35 \times 35$ grid of evaluation points in $\Omega_0$
Example – **Approach 2: Double knot**

Reproduce the domain:
\[ \tilde{\xi} = (0, 0, 0, 0.5, 0.5, 1, 1, 1), \quad \tilde{\eta} = (0, 0, 0, 1, 1, 1) \]

\[
G(\xi, \eta) = \sum_{i=0}^{4} \sum_{j=0}^{2} P_{ij} B_{ij}(\xi, \eta) \quad (\xi, \eta) \in \Omega_0
\]
Example – **Approach 2: Double knot**

- Perform $h$-refinement, considering $m = 2, 4, 8, 16, 32$, $n = 1, 2, 4, 8, 16$, and $\tilde{\mu}^\eta$ with elements equal to one, while $\tilde{\mu}^\xi$ with all elements equal to one except the element corresponding to $\xi = \frac{1}{2}$, that is equal to zero.

- To obtain a basis, we have to neglect two B-splines either with $C^1$ smoothness everywhere or with $C^0$ smoothness only on the boundary of its support, because in this case $\Omega_0$ is subdivided into two subdomains.

The graphs of:

(a) the exact solution

(b) the approximation computed with $m = 8$, $n = 4$

(c) the discrete $L^\infty$-norm of the error computed on a $35 \times 35$ grid of evaluation points in $\Omega_0$
Example

Discrete $L^2$-norm of the error versus interval number per side

<table>
<thead>
<tr>
<th>$(m, n)$</th>
<th>(2,1)</th>
<th>(4,2)</th>
<th>(8,4)</th>
<th>(16,8)</th>
<th>(32,16)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^2$-error, Approach 1</td>
<td>7.1(-1)</td>
<td>4.5(-1)</td>
<td>5.3(-2)</td>
<td>6.4(-3)</td>
<td>6.2(-4)</td>
</tr>
<tr>
<td>$L^2$-error, Approach 2</td>
<td>8.3(-1)</td>
<td>2.2(-1)</td>
<td>1.7(-2)</td>
<td>1.6(-3)</td>
<td>1.8(-4)</td>
</tr>
</tbody>
</table>
Powell-Sabin triangulation generated by a uniform 6-direction mesh [Goodman, 1997-2007; Chui-Jiang, 2003; Bettayeb, 2008; Davydov-Sablonnière, 2010]

\[ \Omega = [0, m_1 h] \times [0, m_2 h] \]

\[ \Delta_{m_1,m_2}^{PS} \text{ uniform 6-direction mesh} \]

\[ \mathcal{S}_3^2(\Omega, \Delta_{m_1,m_2}^{PS}) = \left\{ s \in C^2(\Omega) \mid s|_T \in \mathbb{P}_3(\mathbb{R}^2), \ T \text{ triangle } \in \Delta_{m_1,m_2}^{PS} \right\} \]

space generated by dilation/translation of the multi-box spline \[ \varphi = (\varphi_1, \varphi_2) \]
Multi-box spline $\varphi_1$

- $(m_1 + 1)(m_2 + 1)$ shifts of $\varphi_1$, denoted by
  \[ \varphi_{1,\alpha}(x, y) = \varphi_{1,(i,j)}(x, y) = \varphi_1 \left( \frac{x}{h} - i, \frac{y}{h} - j \right), \]
  with supports centered at the points $c_{\alpha} = c_{i,j} = (ih, jh)$ and $\alpha \in \mathcal{A}_1 = \{(i,j), 0 \leq i \leq m_1, 0 \leq j \leq m_2\}$,
- normalized multi-box spline $\bar{\varphi}_1 = \frac{1}{6} \varphi_1$.

Support and graph of $\bar{\varphi}_1$
Multi-box spline $\varphi_2$

- $(m_1 + 3)(m_2 + 3) - 2$ shifts of $\varphi_2$, denoted by

$$
\varphi_{2, \alpha}(x, y) = \varphi_{2,(i,j)}(x, y) = \varphi_2 \left( \frac{x}{h} - i, \frac{y}{h} - j \right),
$$

with supports centered at the points $c_\alpha = c_{i,j} = (ih, jh)$ and $\alpha \in \mathcal{A}_2 = \{(i, j), -1 \leq i \leq m_1 + 1, -1 \leq j \leq m_2 + 1; (i, j) \neq (m_1 + 1, -1), (-1, m_2 + 1)\}$,

- normalized multi-box spline $\bar{\varphi}_2 = \frac{1}{2} \varphi_2$.

Support and graph of $\bar{\varphi}_2$
$S^2_3(\Omega, \Delta^{PS}_{m_1,m_2})$  [Remogna, 2012]

- spanning functions with supports also outside $\Omega$
- data points inside or on the boundary of the domain
Quasi-interpolants using data points inside or on the boundary of the domain \[\text{[Remogna, 2012]}\]

\[
Qf = \sum_{\alpha \in A_2} [\lambda_{1,\alpha}(f), \lambda_{2,\alpha}(f)] \tilde{\varphi}_\alpha,
\]

where

- \(\tilde{\varphi} = [\tilde{\varphi}_1, \tilde{\varphi}_2]^T\) and \(\tilde{\varphi}_{1,\alpha} \equiv 0\) for \(\alpha \in A_2 \setminus A_1\)
- \(\lambda_{\nu,\alpha}(f) = \sum_{\beta \in F_{\nu,\alpha}} \sigma_{\nu,\alpha}(\beta)f(A_\beta), \ \nu = 1, 2\)
- The finite set of points \(\{A_\beta, \beta \in F_{\nu,\alpha}\}\), \(F_{\nu,\alpha} \subset A = \{(k, l), k = 0, \ldots, m_1, l = 0, \ldots, m_2\}\) lies in some neighbourhood of \(\text{supp } \tilde{\varphi}_{\nu,\alpha} \cap \Omega\)
- \(Q\) exact on the space of polynomials \(\mathbb{P}_3(\mathbb{R}^2)\)
Operator $Q_1$: near-best quasi-interpolant

We obtain the coefficient functionals $\|\lambda_{v,\alpha}\|_\infty$, $v = 1, 2$, by minimizing an upper bound for the QI infinity norm.
Operator $Q_1$: near-best quasi-interpolant

We obtain the coefficient functionals $\|\lambda_{v,\alpha}\|_\infty$, $v = 1, 2$, by minimizing an upper bound for the QI infinity norm.

Operator $Q_2$: quasi-interpolant with superconvergence properties

We impose the superconvergence of the gradient at some specific points of the domain:

- the vertices of squares $A_{k,l} = (kh, lh)$,
- the centers of squares $M_{k,l} = ((k - \frac{1}{2})h, (l - \frac{1}{2})h)$,
- the midpoints $C_{k,l} = ((k - \frac{1}{2})h, lh)$ of horizontal edges $A_{k-1,l}A_{k,l}$,
- the midpoints $D_{k,l} = (kh, (l - \frac{1}{2})h)$ of vertical edges $A_{k,l-1}A_{k,l}$. 
Theorem 1

For the operators $Q_v$, $v = 1, 2$ the following bounds are valid

$$\|Q_1\|_\infty \leq \frac{53}{6} \approx 8.83, \quad \|Q_2\|_\infty \leq \frac{185}{9} \approx 20.56.$$
**Norm and error estimates**

### Theorem 1
For the operators $Q_v$, $v = 1, 2$ the following bounds are valid

$$
\| Q_1 \|_\infty \leq \frac{53}{6} \approx 8.83, \quad \| Q_2 \|_\infty \leq \frac{185}{9} \approx 20.56.
$$

### Theorem 2
Let $f \in C^4(\Omega)$ and $|\gamma| = 0, 1, 2, 3$. Then there exist constants $K_{v, \gamma} > 0$, $v = 1, 2$, such that

$$
\| D^\gamma (f - Q_v f) \|_\infty \leq K_{v, \gamma} h^{4 - |\gamma|} \max_{|\beta| = 4} \| D^\beta f \|_\infty.
$$

where $D^\beta = D^{\beta_1, \beta_2} = \frac{\partial^{|eta|}}{\partial x^{\beta_1} \partial y^{\beta_2}}$, with $\beta_1 + \beta_2 = |\beta|$. 

Example

\[ f(x, y) = \text{Franke's function on } [0, 1]^2 \]
Example – Approximation of the function

$G$: uniform rectangular grid of $300 \times 300$ points in the domain

$Ef = \max_{(u,v) \in G} \lvert f(u, v) - Qf(u, v) \rvert$, for $Q = Q_1, Q_2$

$rf$: numerical convergence order

<table>
<thead>
<tr>
<th>$m_1 = m_2$</th>
<th>$Q_1$</th>
<th>$Q_2$</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$Ef$</td>
<td>$rf$</td>
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Example – *Approximation of the gradient*

\[
\nabla E_f = \max_{(u,v) \in G} \left( \left| \frac{\partial}{\partial x} f(u, v) - \frac{\partial}{\partial x} Qf(u, v) \right| + \left| \frac{\partial}{\partial y} f(u, v) - \frac{\partial}{\partial y} Qf(u, v) \right| \right),
\]

for \( Q = Q_1, Q_2 \)

\( \nabla r_f \): numerical convergence order

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<tr>
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Example – Approximation of the gradient

$G'$: grid of points of superconvergence

$$\nabla Ef = \max_{(u,v) \in G'} \left( |\frac{\partial}{\partial x} f(u, v) - \frac{\partial}{\partial x} Qf(u, v)| + |\frac{\partial}{\partial y} f(u, v) - \frac{\partial}{\partial y} Qf(u, v)| \right) ,$$

for $Q = Q_1, Q_2$

$\nabla rf$: numerical convergence order

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<tr>
<th>$m_1 = m_2$</th>
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Comparison of the two methods

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</tr>
<tr>
<td>$\Downarrow$</td>
</tr>
<tr>
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</tr>
<tr>
<td>loss of independence in the functional construction</td>
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<tr>
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</table>
Solution of integral equations on surfaces in $\mathbb{R}^3$ by spline quasi-interpolation
3D SPLINE SPACES
Partition of the domain $\Omega \subset \mathbb{R}^3$

$$\Omega = [0, m_1 h] \times [0, m_2 h] \times [0, m_3 h] \subset \mathbb{R}^3$$

divided into equal cubes
Trivariate spline space $S^2_4(\Omega, \mathcal{T}_m)$

Partition $\mathcal{T}_m$, $\mathbf{m} = (m_1, m_2, m_3)$

subdivision of a cube into 24 tetrahedra
(type-6 tetrahedral partition)
Trivariate spline space $S_4^2(\Omega, \mathcal{T}_m)$

Partition $\mathcal{T}_m$, $\mathbf{m} = (m_1, m_2, m_3)$

subdivision of a cube into 24 tetrahedra
(type-6 tetrahedral partition)

$\Downarrow$

Spline space $S_4^2(\Omega, \mathcal{T}_m)$
Trivariate spline space $S_4^2(\Omega, T_m)$ [Peters, 1994]

Spline space generated by the scaled translates of the 7-direction box spline $B(x, y, z)$, whose supports overlap with $\Omega$

Support of the 7-direction box spline $B(x, y, z)$:

Truncated rhombic dodecahedron contained in the cube $[-2, 3] \times [-2, 3] \times [0, 5]$ and centered at $\left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}\right)$
Optimal spline quasi-interpolants exact on $\mathbb{P}_3(\mathbb{R}^3)$

$$Q : \mathcal{F} \to S^2_4(\Omega, \mathcal{T}_m)$$

$$f(x, y, z) \approx Qf(x, y, z)$$

of near-best type, i.e. with coefficient functionals obtained by minimizing an upper bound for the QI infinity norm.
Quasi-interpolation nodes

\[\{M_{ijk} = (s_i, t_j, u_k)\}, \text{ with}\]

\[s_0 = 0, \quad s_i = (i - \frac{1}{2})h, \quad 1 \leq i \leq m_1, \quad s_{m_1+1} = m_1 h\]

\[t_0 = 0, \quad t_j = (j - \frac{1}{2})h, \quad 1 \leq j \leq m_2, \quad t_{m_2+1} = m_2 h\]

\[u_0 = 0, \quad u_k = (k - \frac{1}{2})h, \quad 1 \leq k \leq m_3, \quad u_{m_3+1} = m_3 h.\]

inside or on the boundary of \(\Omega\)
Quasi-interpolation nodes

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inside or on the boundary of \( \Omega \)
Optimal spline quasi-interpolants [Dagnino-Lamberti-Remogna, 2012-2014]

- Optimal spline quasi-interpolants exact on $\mathbb{P}_3(\mathbb{R}^3)$

$$Q : \mathcal{F} \rightarrow S_{4}^{2}(\Omega, \mathcal{T}_m)$$

$$f(x, y, z) \approx Qf(x, y, z)$$

of near-best type, i.e. with coefficient functionals obtained by minimizing an upper bound for the QI infinity norm.

- Quasi-interpolation nodes inside or on the boundary of $\Omega$

$$\{M_{ijk} = (s_i, t_j, u_k)\}, \text{ with}$$

$$s_0 = 0, \quad s_i = (i - \frac{1}{2})h, \quad 1 \leq i \leq m_1, \quad s_{m_1+1} = m_1 h$$

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$$u_0 = 0, \quad u_k = (k - \frac{1}{2})h, \quad 1 \leq k \leq m_3, \quad u_{m_3+1} = m_3 h.$$

- Quasi-interpolation nodes also outside $\Omega$ [Remogna, 2011; Dagnino-Lamberti-Remogna, 2013]
Let \( f \in C^r(\Omega) \), \( r = 0, 1, 2, 3 \). Then there exist constants \( \bar{K}_r > 0 \), such that
\[
\| f - Qf \|_\infty \leq \bar{K}_r h^r \omega(D^r f, h).
\]
If in addition \( f \in C^4(\Omega) \) then there exists constant \( \bar{K}_4 > 0 \), such that
\[
\| f - Qf \|_\infty \leq \bar{K}_4 h^4 \max_{|\beta| = 4} \| D^\beta f \|_\infty.
\]
Numerical tests

- Domain = \([a, b]^3\)
- \(h = \frac{b-a}{m}, \ m = m_1 = m_2 = m_3, \ m = 16, 32, 64, 128\)
- \(G = 139 \times 139 \times 139\) uniform grid of evaluation points in \(\Omega\)
- \(E_{Qf} = \max_{u \in G} |f(u) - Qf(u)|, \ Qf \in S_4^2(\Omega, T_m)\)
- \(E_{Rf} = \max_{u \in G} |f(u) - Rf(u)|, \ Rf \in S_2^1(\Omega, P_m)\)

\(R\) is a spline QI in the space \(S_{2,2}^1(\Omega, P_m)\) of trivariate splines on prismatic partitions defined as tensor product of univariate and bivariate \(C^1\) quadratic B-splines. \(R\) is obtained as blending sum of uni and bivariate \(C^1\) quadratic spline QIs \cite{Remogna-Sablonnière2011}

- \(r_Rf, r_Qf\) numerical convergence orders
The Marschner-Lobb function $f_1$

\[ f_1(x, y, z) = \frac{1}{2(1 + \beta_1)} \left( 1 - \sin \frac{\pi z}{2} + \beta_1 \left( 1 + \cos \left( 2\pi \beta_2 \cos \left( \frac{\pi \sqrt{x^2 + y^2}}{2} \right) \right) \right) \right) \]

with $\beta_1 = \frac{1}{4}$ and $\beta_2 = 6$
on the cube $[-1, 1]^3$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$E_Qf_1$</th>
<th>$r_Qf_1$</th>
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The Marschner-Lobb function $f_1$

$$f_1(x, y, z) = \frac{1}{2(1 + \beta_1)} \left( 1 - \sin \frac{\pi z}{2} + \beta_1 \left( 1 + \cos \left( 2\pi \beta_2 \cos \left( \frac{\pi \sqrt{x^2 + y^2}}{2} \right) \right) \right) \right)$$

with $\beta_1 = \frac{1}{4}$ and $\beta_2 = 6$ on the cube $[-1, 1]^3$

The isosurface obtained from (a) $f_1$ and (b) $Qf_1$, with $m = 64$, for the isovalue $\rho = 1/2$
The smooth trivariate test function of Franke type $f_2$

$$f_2(x, y, z) = \frac{1}{2} e^{-10((x - \frac{1}{4})^2 + (y - \frac{1}{4})^2)} + \frac{3}{4} e^{-16((x - \frac{1}{2})^2 + (y - \frac{1}{4})^2 + (z - \frac{1}{4})^2)}$$

$$+ \frac{1}{2} e^{-10((x - \frac{3}{4})^2 + (y - \frac{1}{8})^2 + (z - \frac{1}{2})^2)} - \frac{1}{4} e^{-20((x - \frac{3}{4})^2 + (y - \frac{3}{4})^2)}$$

on the cube $[0, 1]^3$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$E_Qf_2$</th>
<th>$r_Qf_2$</th>
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The smooth trivariate test function of Franke type $f_2$

Isosurfaces of $Qf_2$ for $m = 32$, with different isovalues
Reconstruction of real world data – *CT Head data set*

Gridded volume data set consisting of $256 \times 256 \times 99$ data samples obtained from a CT scan of a cadaver head (courtesy of University of North Carolina)
Isosurfaces of the $C^2$ trivariate quartic spline approximating the *CT Head data set* with isovalues: 

(a) $\rho = 60$, (b) $\rho = 90$, with $\# G \approx 8.6 \cdot 10^6$ evaluation points.
Reconstruction of real world data – MR brain data set

Gridded volume data set of $256 \times 256 \times 99$ data samples obtained from a MR study of head with skull partially removed to reveal brain (courtesy of University of North Carolina)
Reconstruction of real world data – *MR brain data set*

Isosurface of the $C^2$ trivariate quartic spline approximating the *MR brain data set* with isovalue $\rho = 40$, with $\#G \approx 8.6 \cdot 10^6$ evaluation points.
For any function $f \in C(\Omega)$, we consider the evaluation of the integral

$$l(f) = l(f; \Omega) := \int_{\Omega} f(x, y, z) \, dx \, dy \, dz,$$
For any function $f \in C(\Omega)$, we consider the evaluation of the integral

$$I(f) = I(f; \Omega) := \int_{\Omega} f(x, y, z) \, dx \, dy \, dz,$$

by cubature rules defined by

$$I_Q(f) = I(Qf; \Omega) := \sum_{ijk} w_{ijk}^Q f(M_{ijk}),$$

with

- $M_{ijk}$: cubature nodes in $\Omega$. They coincide with the quasi-interpolation nodes
- $w_{ijk}^Q$: cubature weights, linear combinations of $\int_{\Omega \cap \text{supp} B_{ijk}} B_{ijk}$
- the precision degree is 3, because $Q$ is exact on $\mathbb{P}_3(\mathbb{R}^3)$
- if $f \in C^4(\Omega)$, then $| I(f) - I_Q(f) | = O(h^4)$
integration domain: \( \Omega = [0, 1]^3 \)

- \( m_1 = m_2 = m_3 = m, \ h = 1/m \) and \( m = 16, 32, 64, 128 \)

- integrand functions
  - \( f_1(x, y, z) = e^{((x-0.5)^2+(y-0.5)^2+(z-0.5)^2)} \) (smooth test function), \( I(f_1) = 0.7852115962 \)
  - \( f_2 = \frac{27}{8} \sqrt{1 - |2x - 1|} \sqrt{1 - |2y - 1|} \sqrt{1 - |2z - 1|} \) (continuous test function), \( I(f_2) = 1 \)

| \( m \)  | \( |I(f_1) - I_Q(f_1)| \) | \( rf_1 \) | \( |I(f_2) - I_Q(f_2)| \) | \( rf_2 \) |
|--------|-----------------|--------|-----------------|--------|
| 16     | 2.9(-5)         |        | 4.9(-3)         |        |
| 32     | 1.9(-6)         | 3.9    | 2.4(-3)         | 1.1    |
| 64     | 1.3(-7)         | 3.9    | 9.3(-4)         | 1.3    |
| 128    | 8.1(-9)         | 4.0    | 3.5(-4)         | 1.4    |
Systematic method for the construction of families of near-best $C^2$ quartic spline QIs on type-6 tetrahedral partitions of the space
Our references


- C. Dagnino, P. Lamberti, S. Remogna, Near-best $C^2$ quartic spline quasi-interpolants on type-6 tetrahedral partitions of bounded domains, CALCOLO, Published online: 10 October 2014, (2014).


Thank you!