On 1D, 2D and 3D spline quasi-interpolation

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Spline spaces

- $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$
- Δ partition of Ω
- the domain Ω is divided into a finite number of sub-domains D_i, $i = 1, \ldots, N$ by the partition Δ

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Spline space

$$S_k^\mu(\Omega, \Delta) = \left\{ s \in C^\mu(\Omega) \mid s\,|_{D_i} \in \mathbb{P}_k(\mathbb{R}^d),\ i = 1, \ldots, N \right\}$$

$s \in S_k^\mu(\Omega, \Delta)$ is a piecewise polynomial of degree k with μ order continuous (partial) derivatives in Ω

Spline quasi-interpolants on bounded domains

A local spline quasi-interpolant (abbr. QI) of a function f has the general form

$$Q : \mathcal{F} \to S_k^{\mu}(\Omega, \Delta)$$

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- Q exact on the space of polynomials of degree at most r $\mathbb{P}_r(\mathbb{R}^d)$, i.e. $Qp = p, \quad \forall p \in \mathbb{P}_r(\mathbb{R}^d)$, $r \leq k$

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- Q exact on the space of polynomials of degree at most r $\mathbb{P}_r(\mathbb{R}^d)$, i.e. $Qp = p, \quad \forall p \in \mathbb{P}_r(\mathbb{R}^d)$, $r \le k$
- $\|f - Qf\|_{L^p(\Omega)} = O(h^{r+1})$, $1 \le p \le \infty$, f sufficiently smooth function, h maximum of the diameters of elements of Δ

Outline

1D QIs in spline spaces of degree $k = 2, 3$ and smoothness $k - 1$

- Integral equations

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2D - QIs in spaces of quadratic splines on criss-cross triangulations

- Approximation of derivatives
- Error bounds for functions and derivatives
- Construction of NURBS surfaces
- Problems governed by PDEs

- QIs in spaces of C^2 cubic splines on uniform Powell-Sabin triangulations

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- **1D** QIs in spline spaces of degree $k = 2, 3$ and smoothness $k - 1$
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  - QIs in spaces of C^2 cubic splines on uniform Powell-Sabin triangulations
- **3D** QIs in spaces of C^2 quartic splines on uniform type-6 tetrahedral partitions
  - Reconstruction of volumetric data
  - Numerical integration

1D SPLINE SPACES

Univariate spline spaces of degree $k = 2, 3$ and smoothness $k - 1$, $S_k^{k-1}(\Omega, \Delta_n)$

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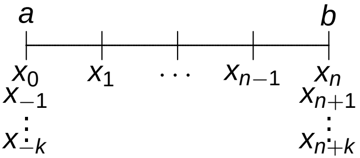

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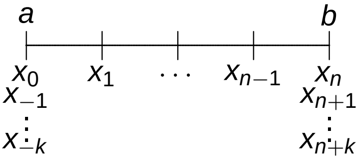

- $\left\{B_j^k(x)\right\}_{j=0}^{n+k-1}$ basis of normalized B-splines defined on Δ_n, with *supp* $B_j^k = [x_{j-k}, x_{j+1}]$, spanning the spline space $S_k^{k-1}(\Omega, \Delta_n)$

Spline quasi-interpolating projectors [Dagnino-Remogna-Sablonnière, 2014]

- Spline quasi-interpolating projectors P_k exact on $\mathbb{P}_k(\mathbb{R})$, $k = 2, 3$.
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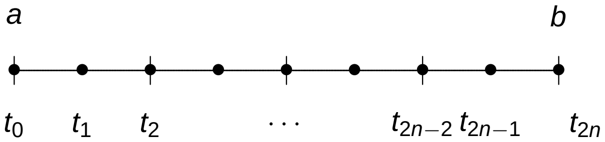
  They are projectors, i.e. $P_k s = s$, $\forall s \in S_k^{k-1}(\Omega, \Delta_n)$
- Quasi-interpolation nodes $\{t_i, i = 0, \ldots, 2n\}$, with

$$
\begin{aligned}
t_{2i} &= x_i, \quad i = 0, \ldots, n \\
t_{2i-1} &= \frac{1}{2}(x_{i-1} + x_i), \quad i = 1, \ldots, n
\end{aligned}
$$



Convergence properties

P_k are uniformly bounded independently of the uniform partition

$$\Downarrow$$

Theorem

For $f^{(k+1)}$ bounded, there holds

$$\|f - P_k f\|_\infty \leq C_k h^{k+1} \|f^{(k+1)}\|_\infty, \quad \text{with } \; C_k = \begin{cases} \frac{7}{24}, & k = 2 \\ \frac{4}{9}, & k = 3 \end{cases}$$

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The operator P_2 is superconvergent at the quasi-interpolation nodes, i.e.
$(f - P_2 f)(t_i) = 0$, for $f \in \mathbb{P}_3(\mathbb{R})$,
$(f - P_2 f)(t_i) = O(h^4)$, for f such that $\|f^{(4)}\|_\infty$ is bounded

Integral equations [Dagnino-Remogna-Sablonnière, 2014]

Linear Fredholm integral equations of the second kind

$$\rho - T\rho = \psi,$$

with

- $T\rho(x) = \int_a^b K(x,y)\rho(y)dy, \quad x \in \Omega$
- $\psi \in C(\Omega)$
- $K \in C(\Omega \times \Omega)$

The four projection methods

1. Galerkin method $\longrightarrow$ approximate equation

$$\rho_n^g - P_k T P_k \rho_n^g = P_k \psi$$

2. Kantorovich method $\longrightarrow$ approximate equation

$$\rho_n^{ka} - P_k T \rho_n^{ka} = \psi$$

3. Sloan's iterated version $\longrightarrow$ approximate equation

$$\rho_n^s - T P_k \rho_n^s = \psi$$

4. Kulkarni's method $\longrightarrow$ approximate equation

$$\rho_n^{ku} - (P_k T + T P_k - P_k T P_k)\rho_n^{ku} = \psi$$

The approximate solutions

The approximate solution for each method is

1. Galerkin method $\rho_n^g = P_k\psi + \sum_{j=0}^{n+k-1} X_j B_j^k$
2. Kantorovich method $\rho_n^{ka} = \psi + \sum_{j=0}^{n+k-1} X_j B_j^k$

We have to solve a linear system and determine the unknowns $\{X_j,\ j = 0,\dots,n+k-1\}$.

3. Sloan's iterated version: it is obtained as an iterate of Galerkin's solution

$$\rho_n^s = \psi + T\rho_n^g \quad \Rightarrow \quad \rho_n^s = \psi + \sum_{j=0}^{n+k-1} (\lambda_j(\psi) + X_j) T B_j^k,$$

$\{X_j,\ j = 0,\dots,n+k-1\}$ determined by Galerkin method

The approximate solutions

4. Kulkarni's method $\rho_n^{ku} = \psi + \sum_{j=0}^{n+k-1} X_j B_j^k + \sum_{i=0}^{n+k-1} Y_i TB_i^k$

The problem has $2(n+k)$ unknowns $\rightarrow$ linear system of $2(n+k)$ equations

The system can be reduced to the solution of **one system** of $n+k$ algebraic equations.

First we determine $\{Y_i,\ j = 0, \ldots, n+k-1\}$ by solving the linear system, then we get $\{X_j,\ j = 0, \ldots, n+k-1\}$

Computation of the solutions

In order to construct the linear systems we have to evaluate different kinds of integrals. For example

- $TB_j(x) = \int_a^b B_j(y)K(x,y)dy \to$ suitable product quadrature formulas (PQF) with B-spline weight functions
- $\int_a^b K(t_j,y)\psi(y)dy \to$ suitable Romberg's quadrature formula
- $T\tilde{B}_i(t_j) = \int_a^b K(t_j,y)\tilde{B}_j(y)dy$, with $\tilde{B}_i(x) = TB_i(x) \to$ suitable Romberg's quadrature formula

$\{t_j, j = 0, \ldots, 2n\}$ are the QI nodes.

Convergence orders of the solutions

Theorem – case $k = 2$

Assume that the solution ρ has a bounded fourth derivative, then there holds

$$\left\|\rho - \rho_n^g\right\|_\infty = O(h^3), \quad \left\|\rho - \rho_n^{ka}\right\|_\infty = O(h^3),$$

$$\|\rho - \rho_n^s\|_\infty = O(h^4), \quad \left\|\rho - \rho_n^{ku}\right\|_\infty = O(h^7)$$

Superconvergence phenomenon at the set of QI nodes $\{t_i, i = 0, \ldots, 2n\}$ in case of Galerkin, Kantorovich and Kulkarni methods

$$\rho(t_i) - \rho_n^g(t_i) = O(h^4),$$
$$\rho(t_i) - \rho_n^{ka}(t_i) = O(h^4),$$
$$\rho(t_i) - \rho_n^{ku}(t_i) = O(h^8),$$

Convergence orders of the solutions

Theorem – case $k = 3$

Assume that the solution ρ has a bounded fourth derivative, then there holds

$$\left\|\rho - \rho_n^{g}\right\|_\infty = O(h^4), \qquad \left\|\rho - \rho_n^{ka}\right\|_\infty = O(h^4),$$

$$\|\rho - \rho_n^{s}\|_\infty = O(h^4 \varepsilon(h)), \quad \lim_{h\to 0} \varepsilon(h) = 0;$$

$$\left\|\rho - \rho_n^{ku}\right\|_\infty = O(h^8).$$

Example

Integral equation $\rho(x) - \int_0^1 K(x,y)\rho(y)dy = \psi(x)$, $x \in [0,1]$,

with

- exact solution $\rho(x) = \exp(-x)\cos(x)$
- kernel $K(x,y) = \exp(xy)$
- function
  $\psi(x) = \exp(-x)\cos(x) + \frac{\exp(x-1)(\cos(1)(1-x)-\sin(1))+x-1}{x^2-2x+2}$

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We compute the maximum absolute error for increasing values of n

$$e_n = \max_{z \in G} |\rho(z) - \rho_n(z)|,$$

where G is a set of 1500 equally spaced points in $[0,1]$.
We also compute the numerical convergence order (NCO).

Example

n	Methods based on P_2							
n	e_n^g	NCO_g	e_n^{ka}	NCO_{ka}	e_n^s	NCO_s	e_n^{ku}	NCO_{ku}
4	2.7(-04)		2.4(-05)		1.3(-04)		2.1(-09)	
8	3.1(-05)	3.1	2.8(-06)	3.1	5.7(-06)	4.5	1.0(-11)	7.7
16	3.9(-06)	3.0	3.4(-07)	3.0	2.8(-07)	4.3	5.4(-14)	7.6
32	4.9(-07)	3.0	4.2(-08)	3.0	1.5(-08)	4.2	1.0(-15)	–
64	6.1(-08)	3.0	5.3(-09)	3.0	8.8(-10)	4.1		–
128	7.6(-09)	3.0	6.5(-10)	3.0	5.3(-11)	4.0		–
256	1.1(-10)	3.0	8.1(-11)	3.0	3.2(-12)	4.0		–

	Methods based on P_3							
n	e_n^g	NCO_g	e_n^{ka}	NCO_{ka}	e_n^s	NCO_s	e_n^{ku}	NCO_{ku}
4	3.1(-05)		3.3(-06)		2.3(-06)		1.2(-09)	
8	2.1(-06)	3.9	2.5(-07)	3.7	7.3(-08)	4.9	5.0(-12)	7.9
16	1.4(-07)	4.0	1.8(-08)	3.8	2.1(-09)	5.1	2.0(-14)	8.0
32	8.7(-09)	4.0	1.2(-09)	3.9	6.0(-11)	5.1	8.0(-16)	–
64	5.5(-10)	4.0	8.1(-11)	3.9	1.8(-12)	5.1		–
128	3.4(-11)	4.0	5.1(-12)	4.0	5.5(-14)	5.0		–
256	2.1(-12)	4.0	3.2(-13)	4.0	2.4(-15)	–		–

Example

We compute the maximum absolute error at the QI nodes $\{t_j, j = 0, \ldots, 2n\}$ for the spline projector P_2

$$es_n = \max_{z \in \{t_j, j=0,\ldots,2n\}} |\rho(z) - \rho_n(z)|$$

n	es_n^g	NCO_g	es_n^{ka}	NCO_{ka}	es_n^{ku}	NCO_{ku}
4	1.3(-04)		4.8(-06)		1.3(-09)	
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Superconvergence

Work in progress

Spline quasi-interpolation for the solution of:

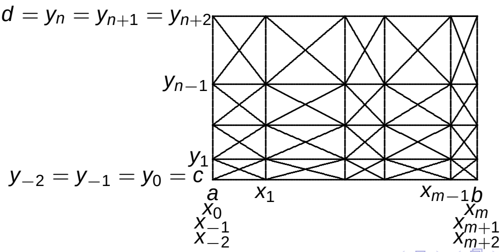
- integral equations with Green's function kernels
- weakly singular integral equations
- line integral equations

2D SPLINE SPACES

Bivariate quadratic spline spaces on criss-cross triangulations Δ_{mn}: $S_2^1(\Omega, \Delta_{mn})$

Let:

- $\Omega = [a, b] \times [c, d]$ be a rectangular domain
- m, n be positive integers
- $\bar{x} = (x_i)_{i=-2}^{m+2}$, where $x_{-2} = x_{-1} = x_0 = a < x_1 < \ldots < x_{m-1} < b = x_m = x_{m+1} = x_{m+2}$
- $\bar{y} = (y_j)_{j=-2}^{n+2}$, where $y_{-2} = y_{-1} = y_0 = c < y_1 < \ldots < y_{n-1} < d = y_n = y_{n+1} = y_{n+2}$
- Δ_{mn} be the corresponding criss-cross triangulation



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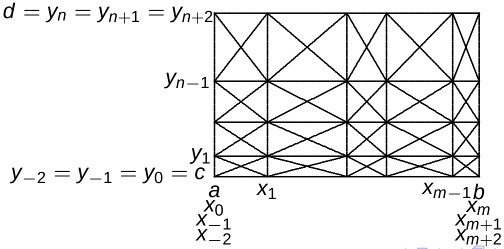
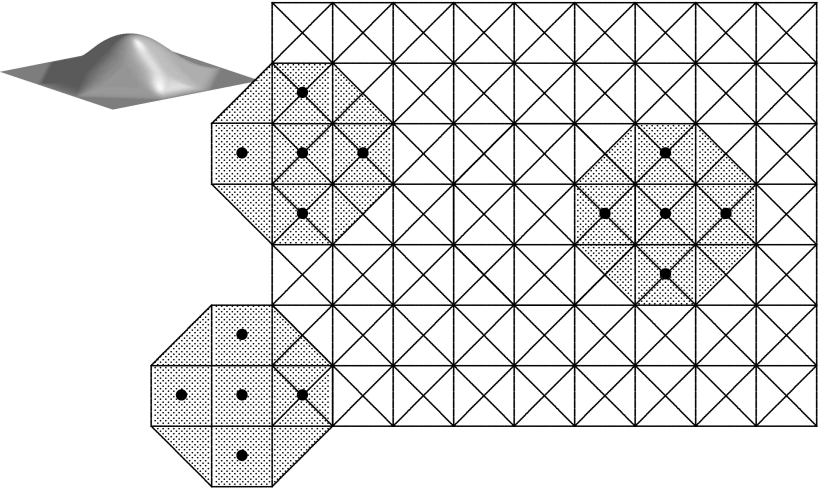


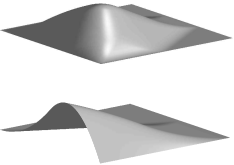
$S_2^1(\Omega, \Delta_{mn})$

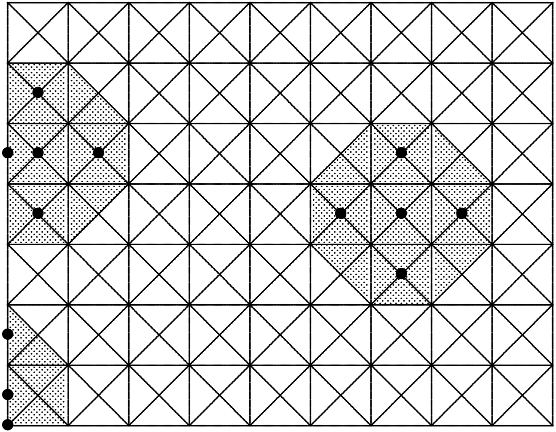
Example: $S_2^1(\Omega, \Delta_{mn})$ [BOX SPLINES: de Boor, Chui, Dagnino, Dahmen, Höllig, Lyche, Micchelli, Sablonnière, Schumaker, Wang, ...]



- classical spanning functions
- data points also outside the domain

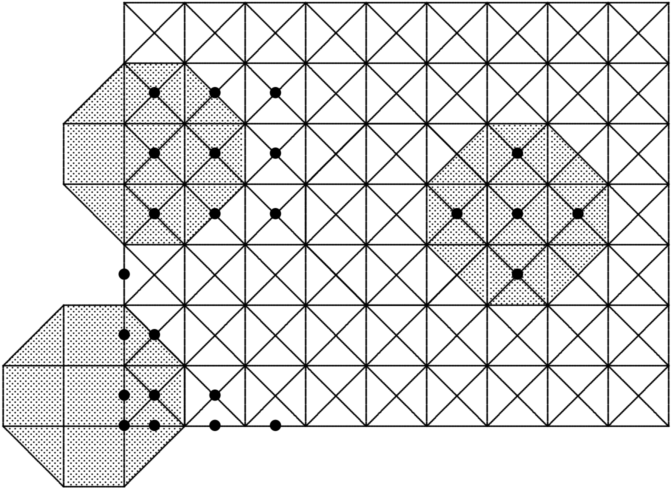
First method [Sablonnière, Dagnino, Demichelis, Lamberti, Remogna, ...]





- spanning functions with support completely in the domain
- data points inside or on the boundary of the domain

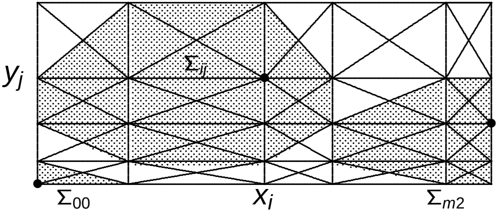
Second method [Remogna, 2010]

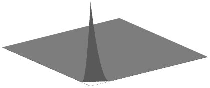


- classical spanning functions
- data points inside or on the boundary of the domain

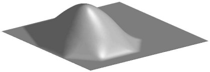
Spanning set $\mathcal{B}$ [Sablonnière, 2003]

$\mathcal{B} = \{B_{ij}\}_{i=0,j=0}^{m+1\ n+1}$ collection of $(m+2)(n+2)$ quadratic B-splines, with support Σ_{ij}, spanning $S_2^1(\Omega, \Delta_{mn})$

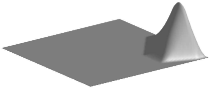




B_{00}



B_{ij}



B_{m2}

Dimension and basis

$$dim\ S_2^1(\Omega, \Delta_{mn}) = (m+2)(n+2) - 1$$

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$$\Downarrow$$

The B-splines in the spanning set $\mathcal{B}$ are linearly dependent

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$$\Downarrow$$

The B-splines in the spanning set $\mathcal{B}$ are linearly dependent

In order to obtain a basis, we have to remove one B-spline from $\mathcal{B}$, with C^1 smoothness everywhere or with C^0 smoothness on the boundary of its support

Optimal spline quasi-interpolants [Sablonnière, 2003; Dagnino-Lamberti, 2008]

- Optimal spline quasi-interpolants exact on $\mathbb{P}_2(\mathbb{R}^2)$

$$Q : \mathcal{F} \to S_2^1(\Omega, \Delta_{mn})$$
$$f(x, y) \approx Qf(x, y)$$

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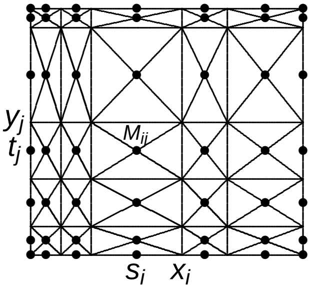
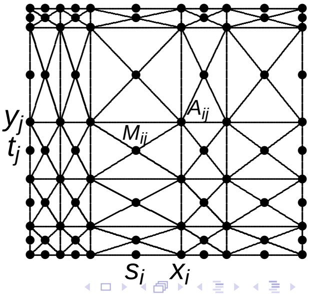
- Quasi-interpolation nodes:
  $\{M_{ij} = (s_i, t_j)\}$, $s_i = \frac{x_{i-1}+x_i}{2}$, $t_j = \frac{y_{j-1}+y_j}{2}$
  $\{A_{ij} = (x_i, y_j)\}$





Approximation of partial derivatives [Dagnino-Remogna-Sablonnière, 2013]

$$D^\alpha f(\overline{x},\overline{y}) \approx D^\alpha Qf(\overline{x},\overline{y})$$

with:

- $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $0 \leq |\alpha| \leq 2$
- $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial^{\alpha_1} x \partial^{\alpha_2} y}$
- $(\overline{x},\overline{y}) \in T$ triangle in Δ_{mn} for $|\alpha| = 0, 1$
- $(\overline{x},\overline{y}) \in int(T)$, T triangle in Δ_{mn} for $|\alpha| = 2$

Error $\|D^\alpha(f - Qf)\|_\infty$ [Dagnino-Remogna-Sablonnière, 2013]

Let $f \in C^\nu(\Omega)$, with $0 \le |\alpha| \le \nu \le 2$, $|\alpha| = 0, 1, 2$, then

$$\|D^\alpha(f - Qf)\|_{\infty,T} \le \overline{C}_{|\alpha|,\nu} \left(\frac{h_T}{h_T^*}\right)^{|\alpha|} h_T^{\nu-|\alpha|}\omega\left(D^\nu f, \frac{h_T}{2}, \Omega\right).$$

If, in addition, $f \in C^3(\Omega)$, then

$$\|D^\alpha(f - Qf)\|_{\infty,T} \le \overline{C}_{|\alpha|,3} \left(\frac{h_T}{h_T^*}\right)^{|\alpha|} h_T^{3-|\alpha|} \left\|D^3 f\right\|_\Omega,$$

with $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$, $h_T = \max\{h_i, k_j\}$, $h_T^* = \min\{h_i, k_j\}$.

Global results only for $|\alpha| = 0, 1$.

Error $\|D^\alpha(f - Qf)\|_\infty$ [Dagnino-Remogna-Sablonnière, 2013]

- For $|\alpha| = 0$, the error bounds are independent of the mesh ratios $\left(\frac{h_T}{h_T^*}\right) \Rightarrow Qf \to f$ in T as $h_T \to 0$.

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  Such ratios are bounded for example in case of uniform triangulation or γ-quasi uniform partitions ($0 < h/h^* \leq \gamma$, $\gamma > 1$ constant).

Example

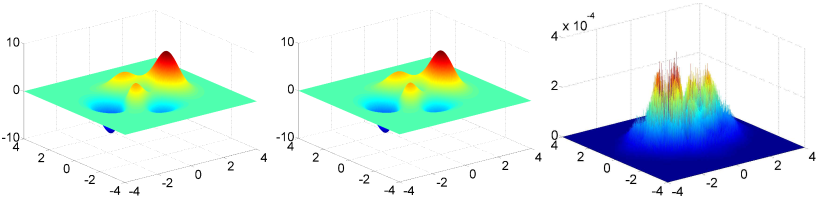
- Test function on $\Omega = [-4, 4]^2$

$$f(x, y) = 3(1-x)^2 e^{(-x^2-(y+1)^2)} - 10\left(\frac{x}{5} - x^3 - y^5\right) e^{(-x^2-y^2)} - \frac{1}{3} e^{(-(x+1)^2-y^2)}$$

- G is a 300×300 uniform rectangular grid of evaluation points in Ω
- $f\_error = \max_{(u,v)\in G} |(f - Qf)(u, v)|$
- $D^{(\alpha_1,\alpha_2)} f\_error = \max_{(u,v)\in G} |D^{(\alpha_1,\alpha_2)}(f - Qf)(u, v)|, \ |\alpha| = 1$

Example

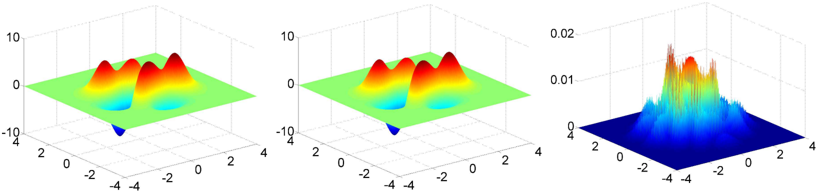
The graphs of f, Qf and $|f - Qf|$ computed on the grid G, considering a uniform triangulation with $m = n = 128$



f $\qquad$ Qf $\qquad$ $|f - Qf|$

Example

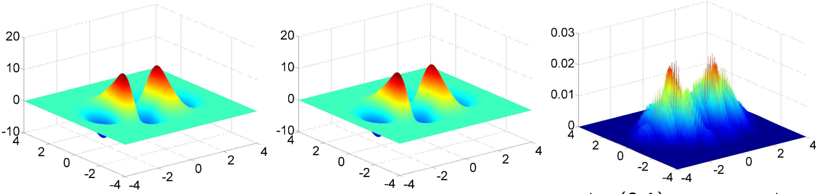
The graphs of $D^{(1,0)}f$, $D^{(1,0)}Qf$ and $\left|D^{(1,0)}(f - Qf)\right|$ computed on the grid G, considering a uniform triangulation with $m = n = 128$



$D^{(1,0)}f$ $D^{(1,0)}Qf$ $\left|D^{(1,0)}(f - Qf)\right|$

Example

The graphs of $D^{(0,1)}f$, $D^{(0,1)}Qf$ and $\left|D^{(0,1)}(f - Qf)\right|$ computed on the grid G, considering a uniform triangulation with $m = n = 128$

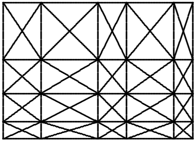


$D^{(0,1)}f$ $D^{(0,1)}Qf$ $\left|D^{(0,1)}(f - Qf)\right|$

Quadratic spline space $S_2^{(\bar{\mu}^x,\bar{\mu}^y)}(\Omega, \Delta_{mn})$ [Dagnino-Lamberti-Remogna, 2012]

Let $S_2^{(\bar{\mu}^x,\bar{\mu}^y)}(\Omega, \Delta_{mn})$ be the space of bivariate quadratic piecewise polynomials on Δ_{mn}, where

- $\bar{\mu}^x = (\mu_i^x)_{i=1}^{m-1}$ and $\bar{\mu}^y = (\mu_j^y)_{j=1}^{n-1}$ are vectors whose elements can be 1,0,-1 and denote the smoothness C^1, C^0, C^{-1}, respectively, across the inner grid lines $x - x_i = 0, i = 1, \ldots, m-1$ and $y - y_j = 0, j = 1, \ldots, n-1$,
- while the smoothness across all oblique mesh segments[1] is C^1



[1] We call *mesh segments* the line segments that form the boundary of each triangular cell of Δ_{mn}.

Dimension

Given $s \in S_2^{(\bar{\mu}^x,\bar{\mu}^y)}(\Omega, \Delta_{mn})$, we denote by

- L_x^0 (L_x^{-1}) the number of vertical grid lines $x - x_i = 0$, $i = 1, \ldots, m-1$ across which s has C^0 (C^{-1}) smoothness
- L_y^0 (L_y^{-1}) the number of horizontal grid lines $y - y_j = 0$, $j = 1, \ldots, n-1$ across which s has C^0 (C^{-1}) smoothness

$$\Downarrow$$

$$dim\ S_2^{(\bar{\mu}^x,\bar{\mu}^y)}(\Omega, \Delta_{mn}) = d_1 + d_2 + d_3,$$

where

$$\begin{aligned} d_1 &= (m+2)(n+2) - 1 \\ d_2 &= (n+1)L_x^0 + (m+1)L_y^0 \\ d_3 &= (2n+3+L_y^0+L_y^{-1})L_x^{-1} + (2m+3+L_x^0+L_x^{-1})L_y^{-1} + L_x^{-1}L_y^{-1} \end{aligned}$$

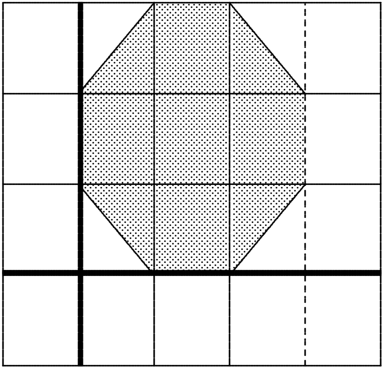
Spanning set $\mathcal{B}$

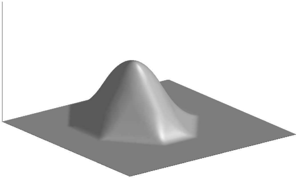
$\mathcal{B} = \{B_{ij}\}_{i=0,j=0}^{M-1\ N-1}$ collection of $M \cdot N$ quadratic B-splines with multiple knots, spanning $S_2^{(\bar{\mu}^x, \bar{\mu}^y)}(\Omega, \Delta_{mn})$, where

- $M = 2 + \sum_{i=1}^{m-1} m_i^x$, $N = 2 + \sum_{j=1}^{n-1} m_j^y$
- $\bar{m}^x = (m_i^x)_{i=1}^{m-1}$, $m_i^x = 2 - \mu_i^x$, $\bar{m}^y = (m_j^y)_{j=1}^{n-1}$, $m_j^y = 2 - \mu_j^y$ vectors of knot multiplicity

B_{ij} smoothness and support, containing multiple knots, change as the number of triangular cells on which the function is nonzero is reduced

- a thick line corresponds to a double knot
- a dotted line corresponds to a triple knot





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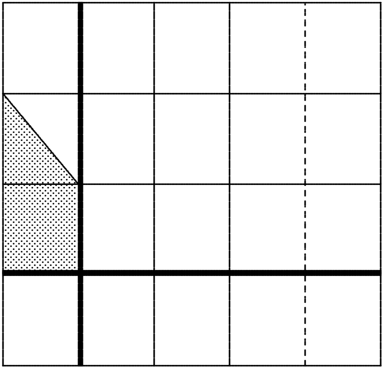

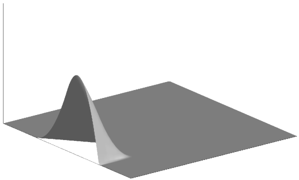

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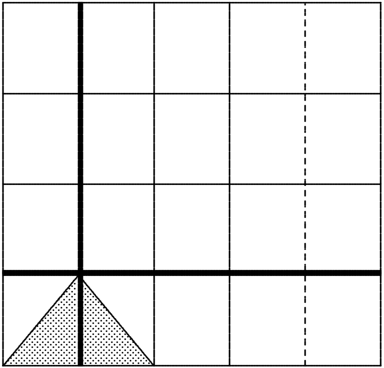

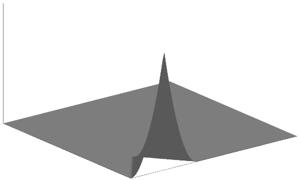

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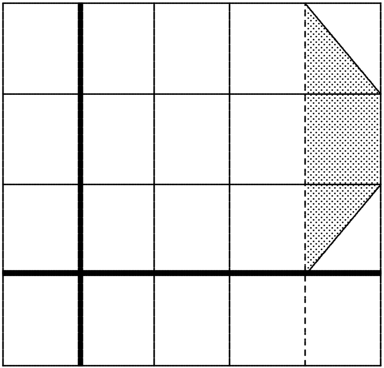

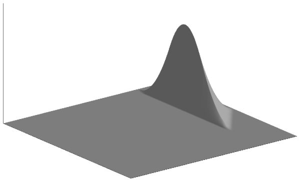

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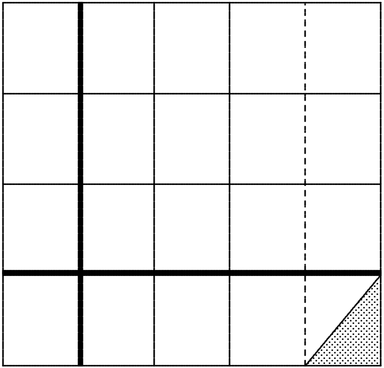

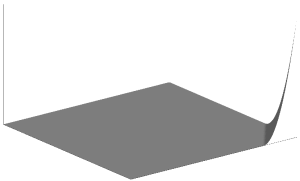

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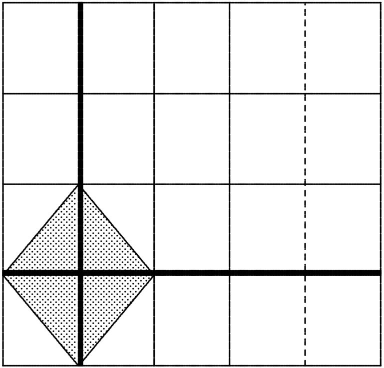

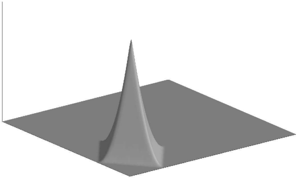

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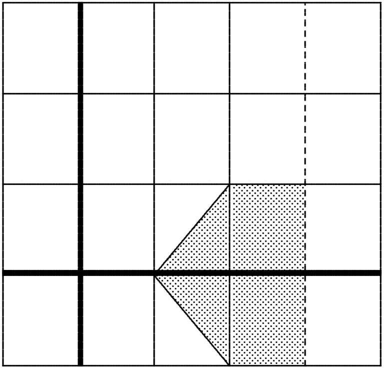

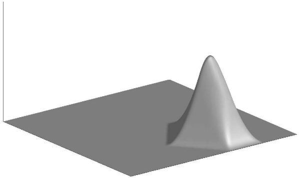

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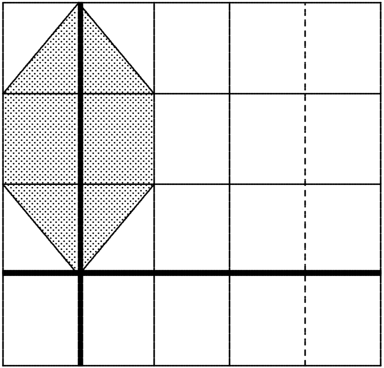

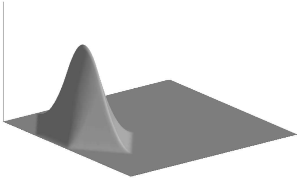

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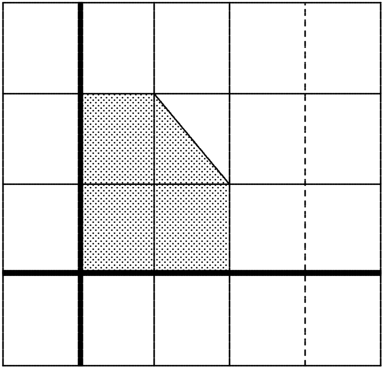

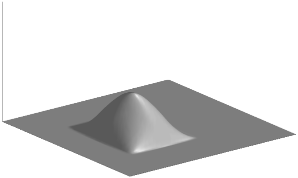

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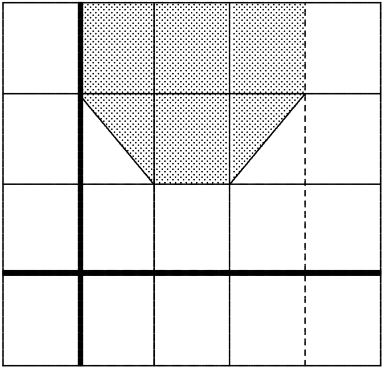

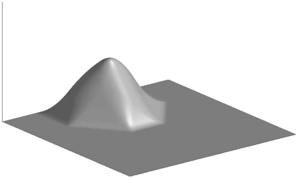

B-spline basis

$$\sharp\mathcal{B} = M \cdot N > dim\ \mathcal{S}_2^{(\bar{\mu}^{\xi},\bar{\mu}^{\eta})}(\mathcal{T}_{mn})$$

i.e. the elements of $\mathcal{B}$ are linearly dependent.

B-spline basis

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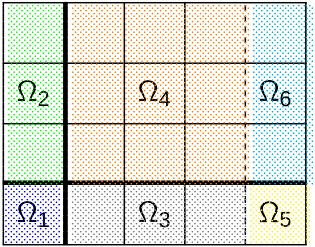
i.e. the elements of $\mathcal{B}$ are linearly dependent.
In order to obtain a basis, we have to remove one B-spline for each subdomain, with C^1 smoothness everywhere or with C^0 smoothness only on the boundary of its support



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Applications to NURBS surfaces

We provide:

- bidirectional net of control points $(\mathbf{P}_{ij})$, $\mathbf{P}_{ij} \in \mathbb{R}^3$
- real positive weights (w_{ij})
- suitable knot vectors $\bar{x}$ and $\bar{y}$ in the parametric domain Ω

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and we define the NURBS surface

$$\mathbf{S}(x,y) = \frac{\sum_{ij} w_{ij}\, \mathbf{P}_{ij}\, B_{ij}(x,y)}{\sum_{ij}\, w_{ij}\, B_{ij}(x,y)} = \sum_{ij} \mathbf{P}_{ij}\, R_{ij}(x,y), \quad (x,y) \in \Omega$$

with

$$R_{ij}(x,y) = \frac{w_{ij}\, \mathbf{P}_{ij}\, B_{ij}(x,y)}{\sum_{rs}\, w_{rs}\, B_{rs}(x,y)}$$

- The B-splines in $\mathcal{B}$ are non negative and satisfy the property of unity partition $\Rightarrow$ $\mathbf{S}(x, y)$ has convex hull and affine transformation invariance properties.
- $\bar{x}$ and $\bar{y}$ have simple knots $\Rightarrow$ $\mathbf{S}(x, y)$ is C^1.
- B-spline locality property $\Rightarrow$ $\mathbf{S}(x, y)$ interpolates the control points $\mathbf{P}_{ij}$ whose pre-images are the corner points of each subdomain.
- In case $w_{ij} = w, \ \forall (i, j)$, then $\mathbf{S}(x, y)$ is a B-spline surface.

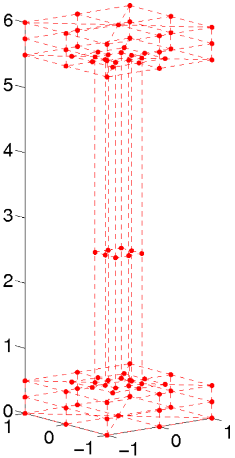
  If, in addition, we consider a functional parametrization, $\mathbf{S}(x, y)$ is the spline function defined by the well known bivariate “variation diminishing” operator, reproducing bilinear functions.

Example

We want to reconstruct a bar bell by a suitable quadratic NURBS surface

Example

We consider the bidirectional net of control points $(\mathbf{P}_{ij})_{i=0,j=0}^{8\ \ 12}$,



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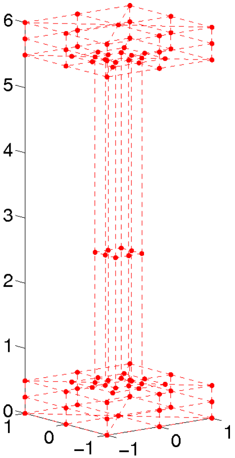

and the weights $(w_{ij})_{i=0,j=0}^{8\ \ 12}$

$$\begin{aligned} w_{00} = w_{10} = w_{20} = w_{30} = w_{40} = \\ = w_{50} = w_{60} = w_{70} = w_{80} = 1, \end{aligned}$$

$$\begin{aligned} w_{0j} = w_{2j} = w_{4j} = w_{6j} = w_{8j} = 1, \\ w_{1j} = w_{3j} = w_{5j} = w_{7j} = \frac{\sqrt{2}}{2},\ j = 1, \ldots, 11, \end{aligned}$$

$$\begin{aligned} w_{0,12} = w_{1,12} = w_{2,12} = w_{3,12} = w_{4,12} = \\ = w_{5,12} = w_{6,12} = w_{7,12} = w_{8,12} = 1. \end{aligned}$$

Example

If we assume:

- $\Omega = [0,1] \times [0,1]$
- $\bar{x} = (0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, 1, 1, 1), \bar{y} = (0, 0, 0, \frac{1}{7}, \frac{2}{7}, \frac{2}{7}, \frac{3}{7}, \frac{3}{7}, \frac{4}{7}, \frac{4}{7}, \frac{5}{7}, \frac{5}{7}, \frac{6}{7}, 1, 1, 1)$

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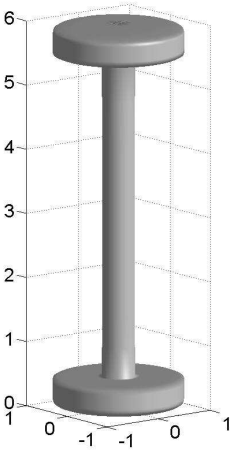
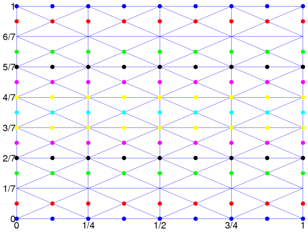


then we can model the object by the C^0 NURBS surface

$$\mathbf{S}(x,y) = \frac{\sum_{i=0}^{8}\sum_{j=0}^{12} w_{ij}\,\mathbf{P}_{ij}\,B_{ij}(x,y)}{\sum_{i=0}^{8}\sum_{j=0}^{12} w_{ij}\,B_{ij}(x,y)}$$



Parametric domain Ω

Applications to elliptic diffusion-type problems with mixed boundary conditions [Cravero-Dagnino-Remogna, 2012]

Let:

- $\Omega \subset \mathbb{R}^2$ be an open, bounded and Lipschitz domain
- $\emptyset \subseteq \Gamma_D, \Gamma_N \subseteq \partial\Omega$, $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$

$$\begin{cases} -\nabla \cdot (K \nabla u) = f, & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}_K} = g_N & \text{on } \Gamma_N, \quad \text{(Neumann conditions)} \\ u = g, & \text{on } \Gamma_D \quad \text{(Dirichlet conditions)} \end{cases}$$

where

- $K \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix
- $\mathbf{n}_K = K\mathbf{n}$ is the outward conormal vector on Γ_N
- $f \in L^2(\Omega)$
- $g_N \in L^2(\Gamma_N)$
- $g \in H^{1/2}(\Gamma_D)$ (g is the trace on Γ_D of function in $H^1(\Omega) = \{v \in L^2(\Omega) : \; D^\alpha v \in L^2(\Omega), |\alpha| \leq 1\}$)

Reconstruction of the domain

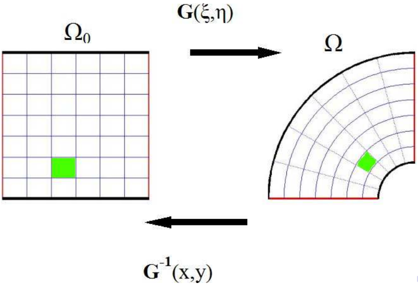
Ω domain in $\mathbb{R}^2$, exactly described through the parametrization

$$\mathbf{G} : \Omega_0 \to \bar{\Omega}, \qquad \mathbf{G}(\xi,\eta) = \begin{pmatrix} x \\ y \end{pmatrix}$$

expressed as quadratic NURBS surface

$$\mathbf{G}(\xi,\eta) = \sum_{ij} \mathbf{P}_{ij}\, R_{ij}(\xi,\eta),$$

with $\{\mathbf{P}_{ij}\}$, $\mathbf{P}_{ij} \in \mathbb{R}^2$, bidirectional net of control points



Numerical solution

Weak formulation

$$\Downarrow$$

Galerkin method
(discretized problem depending on the parameter $h > 0$)

Numerical solution

Weak formulation

$\Downarrow$

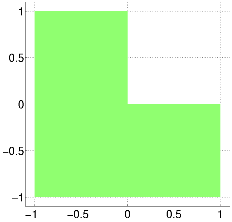
Galerkin method
(discretized problem depending on the parameter $h > 0$)

u_h approximation of the solution u: $u_h = \sum_{ij} q_{ij}(R_{ij} \circ \mathbf{G}^{-1})$

q_{ij} to be determined by solving a linear system

Example

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D = \partial\Omega, \end{cases}$$

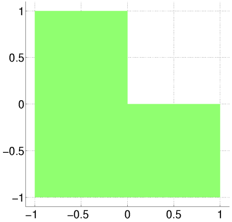


exact solution
$u(x, y) = \sin(\pi x) \sin(\pi y)$

- Reproduce the domain $\rightarrow$ introduce a discontinuity in the first derivative and create the corners $\rightarrow$ two approaches:
  1. place two control points at the same location in physical space;
  2. use suitable double knots in the knot vectors.

Example

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D = \partial\Omega, \end{cases}$$



exact solution
$u(x, y) = \sin(\pi x)\sin(\pi y)$

- Reproduce the domain $\rightarrow$ introduce a discontinuity in the first derivative and create the corners $\rightarrow$ two approaches:
  1. place two control points at the same location in physical space;
  2. use suitable double knots in the knot vectors.

$$\Downarrow$$

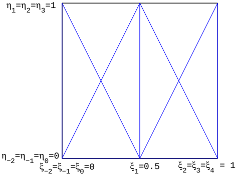
In the first case, we ensure that the basis has C^1 continuity throughout the interior of the domain

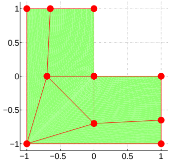
Example – *Approach 1: Double control point*

Reproduce the domain:

$\bar{\xi} = (0, 0, 0, 0.5, 1, 1, 1)$, $\bar{\eta} = (0, 0, 0, 1, 1, 1)$

$$\mathbf{G}(\xi, \eta) = \sum_{i=0}^{3} \sum_{j=0}^{2} \mathbf{P}_{ij}\, B_{ij}(\xi, \eta) \quad (\xi, \eta) \in \Omega_0$$

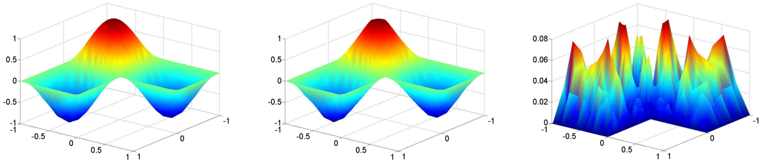




i	$\mathbf{P}_{i0}$	$\mathbf{P}_{i1}$	$\mathbf{P}_{i2}$
0	$(-1, 1)$	$(-0.65, 1)$	$(0, 1)$
1	$(-1, -1)$	$(-0.7, 0)$	$(0, 0)$
2	$(-1, -1)$	$(0, -0.7)$	$(0, 0)$
3	$(1, -1)$	$(1, -0.65)$	$(1, 0)$

Example – *Approach 1: Double control point*

- Perform h-refinement, considering $m = 2, 4, 8, 16, 32$, $n = 1, 2, 4, 8, 16$, and smoothness vectors $\bar{\mu}^{\xi}$, $\bar{\mu}^{\eta}$ with elements equal to one
- To obtain a basis, we have to neglect one B-spline either with C^1 smoothness everywhere or with C^0 smoothness on the boundary of its support



(a) (b) (c)

The graphs of

(a) the exact solution

(b) the approximation computed with $m = 8$, $n = 4$

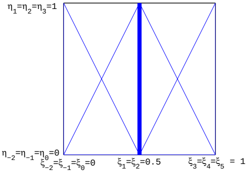
(c) the discrete L^{∞}-norm of the error computed on a 35×35 grid of evaluation points in Ω_0

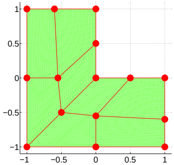
Example – *Approach 2: Double knot*

Reproduce the domain:

$\bar{\xi} = (0, 0, 0, {\color{red}0.5}, {\color{red}0.5}, 1, 1, 1)$, $\bar{\eta} = (0, 0, 0, 1, 1, 1)$

$$\mathbf{G}(\xi, \eta) = \sum_{i=0}^{4} \sum_{j=0}^{2} \mathbf{P}_{ij} \, B_{ij}(\xi, \eta) \quad (\xi, \eta) \in \Omega_0$$

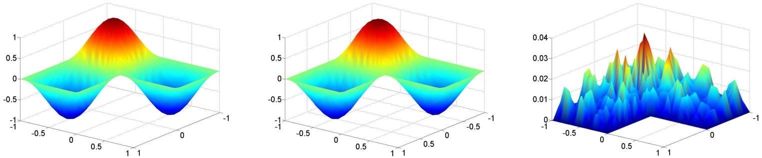




i	$\mathbf{P}_{i0}$	$\mathbf{P}_{i1}$	$\mathbf{P}_{i2}$
0	$(-1, 1)$	$(-0.6, 1)$	$(0, 1)$
1	$(-1, 0)$	$(-0.55, 0)$	$(0, 0.5)$
2	$(-1, -1)$	$(-0.5, -0.5)$	$(0, 0)$
3	$(0, -1)$	$(0, -0.55)$	$(0.5, 0)$
4	$(1, -1)$	$(1, -0.6)$	$(1, 0)$

Example – *Approach 2: Double knot*

- Perform h-refinement, considering $m = 2, 4, 8, 16, 32$, $n = 1, 2, 4, 8, 16$, and $\bar{\mu}^{\eta}$ with elements equal to one, while $\bar{\mu}^{\xi}$ with all elements equal to one except the element corresponding to $\xi = \frac{1}{2}$, that is equal to zero
- To obtain a basis, we have to neglect two B-splines either with C^1 smoothness everywhere or with C^0 smoothness only on the boundary of its support, because in this case Ω_0 is subdivided into two subdomains



(*a*) (*b*) (*c*)

The graphs of

(*a*) the exact solution

(*b*) the approximation computed with $m = 8$, $n = 4$

(*c*) the discrete L^{∞}-norm of the error computed on a 35×35 grid of evaluation points in Ω_0

Example

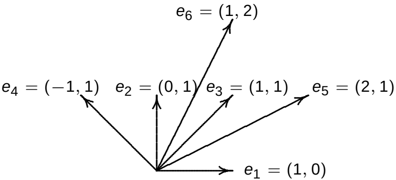
Discrete L^2-norm of the error versus interval number per side

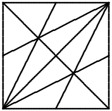
(m, n)	(2,1)	(4,2)	(8,4)	(16,8)	(32,16)
L^2-error, Approach 1	7.1(-1)	4.5(-1)	5.3(-2)	6.4(-3)	6.2(-4)
L^2-error, Approach 2	8.3(-1)	2.2(-1)	1.7(-2)	1.6(-3)	1.8(-4)

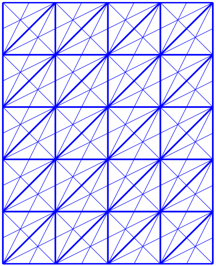
Powell-Sabin triangulation generated by a uniform 6-direction mesh [Goodman, 1997-2007; Chui-Jiang, 2003; Bettayeb, 2008; Davydov-Sablonnière, 2010]

$$\Omega = [0, m_1 h] \times [0, m_2 h]$$

$\Delta^{PS}_{m_1,m_2}$ uniform 6-direction mesh







$$S^2_3(\Omega, \Delta^{PS}_{m_1,m_2}) = \left\{ s \in C^2(\Omega) \mid s\,|_T \in \mathbb{P}_3(\mathbb{R}^2),\ T \text{ triangle } \in \Delta^{PS}_{m_1,m_2} \right\}$$

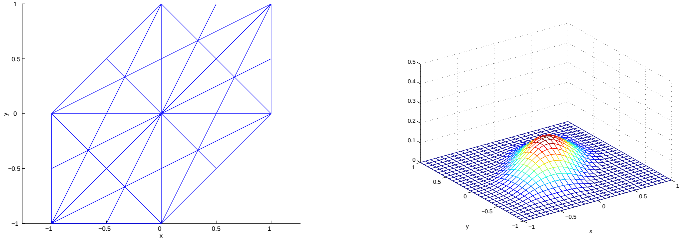
space generated by dilation/translation of the multi-box spline $\varphi = (\varphi_1, \varphi_2)$

Multi-box spline φ_1

- $(m_1+1)(m_2+1)$ shifts of φ_1, denoted by

$$\varphi_{1,\alpha}(x,y) = \varphi_{1,(i,j)}(x,y) = \varphi_1\left(\frac{x}{h} - i, \frac{y}{h} - j\right),$$

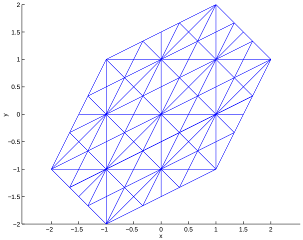
  with supports centered at the points $c_\alpha = c_{i,j} = (ih, jh)$ and $\alpha \in \mathcal{A}_1 = \{(i,j),\ 0 \le i \le m_1,\ 0 \le j \le m_2\}$,
- normalized multi-box spline $\bar{\varphi}_1 = \frac{1}{6}\varphi_1$.

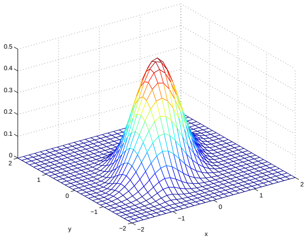


Support and graph of $\bar{\varphi}_1$

Multi-box spline φ_2

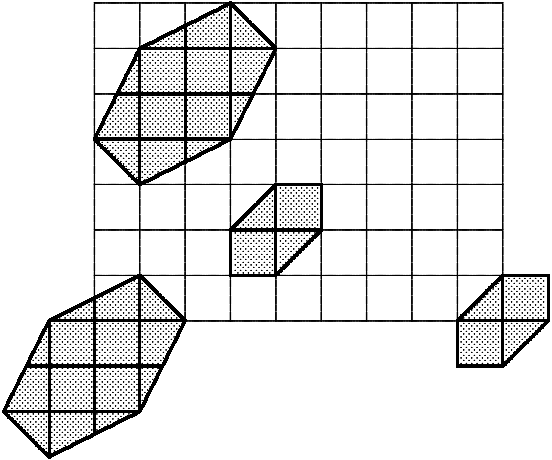
- $(m_1+3)(m_2+3)-2$ shifts of φ_2, denoted by
$$\varphi_{2,\alpha}(x,y)=\varphi_{2,(i,j)}(x,y)=\varphi_2\left(\frac{x}{h}-i,\frac{y}{h}-j\right),$$
with supports centered at the points $c_\alpha=c_{i,j}=(ih,jh)$ and $\alpha\in\mathcal{A}_2=\{(i,j),\ -1\le i\le m_1+1,\ -1\le j\le m_2+1;\ (i,j)\neq(m_1+1,-1),(-1,m_2+1)\}$,
- normalized multi-box spline $\bar{\varphi}_2=\frac{1}{2}\varphi_2$.





Support and graph of $\bar{\varphi}_2$

$S_3^2(\Omega, \Delta^{PS}_{m_1,m_2})$ [Remogna, 2012]



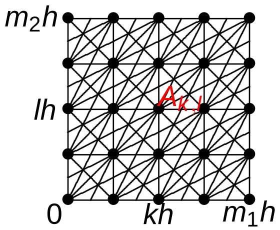
- spanning functions with supports also outside Ω
- data points inside or on the boundary of the domain

Quasi-interpolants using data points inside or on the boundary of the domain [Remogna, 2012]

$$Qf = \sum_{\alpha \in \mathcal{A}_2} [\lambda_{1,\alpha}(f), \lambda_{2,\alpha}(f)]\bar{\varphi}_\alpha,$$

where

- $\bar{\varphi} = [\bar{\varphi}_1, \bar{\varphi}_2]^T$ and $\bar{\varphi}_{1,\alpha} \equiv 0$ for $\alpha \in \mathcal{A}_2 \backslash \mathcal{A}_1$
- $\lambda_{v,\alpha}(f) = \sum_{\beta \in F_{v,\alpha}} \sigma_{v,\alpha}(\beta) f(A_\beta), \quad v = 1, 2$
- The finite set of points $\{A_\beta, \ \beta \in F_{v,\alpha}\}$, $F_{v,\alpha} \subset \mathcal{A} = \{(k,l), k = 0, \ldots, m_1, l = 0, \ldots, m_2\}$ lies in some neighbourhood of supp $\bar{\varphi}_{v,\alpha} \cap \Omega$



- Q exact on the space of polynomials $\mathbb{P}_3(\mathbb{R}^2)$

Operator Q_1: near-best quasi-interpolant

We obtain the coefficient functionals $\|\lambda_{v,\alpha}\|_\infty$, $v = 1, 2$, by minimizing an upper bound for the QI infinity norm

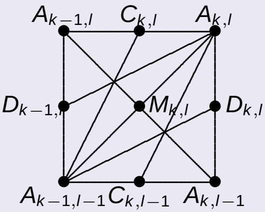
Operator Q_1: near-best quasi-interpolant

We obtain the coefficient functionals $\|\lambda_{v,\alpha}\|_\infty$, $v = 1, 2$, by minimizing an upper bound for the QI infinity norm

Operator Q_2: quasi-interpolant with superconvergence properties

We impose the superconvergence of the gradient at some specific points of the domain

- the vertices of squares $A_{k,l} = (kh, lh)$,
- the centers of squares $M_{k,l} = ((k - \frac{1}{2})h, (l - \frac{1}{2})h)$,
- the midpoints $C_{k,l} = ((k - \frac{1}{2})h, lh)$ of horizontal edges $A_{k-1,l}A_{k,l}$,
- the midpoints $D_{k,l} = (kh, (l - \frac{1}{2})h)$ of vertical edges $A_{k,l-1}A_{k,l}$,



Norm and error estimates

Theorem 1

For the operators Q_v, $v = 1, 2$ the following bounds are valid

$$\|Q_1\|_\infty \leq \frac{53}{6} \approx 8.83, \quad \|Q_2\|_\infty \leq \frac{185}{9} \approx 20.56.$$

Norm and error estimates

Theorem 1

For the operators Q_v, $v = 1, 2$ the following bounds are valid

$$\|Q_1\|_\infty \leq \frac{53}{6} \approx 8.83, \quad \|Q_2\|_\infty \leq \frac{185}{9} \approx 20.56.$$

Theorem 2

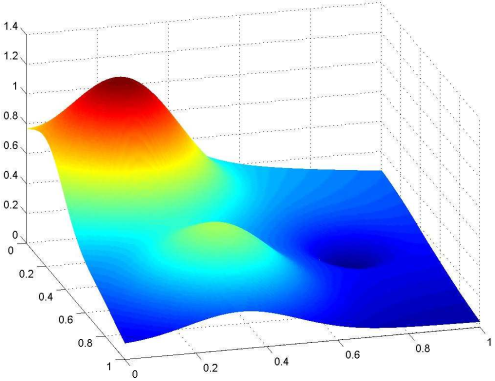
Let $f \in C^4(\Omega)$ and $|\gamma| = 0, 1, 2, 3$. Then there exist constants $K_{v,\gamma} > 0$, $v = 1, 2$, such that

$$\|D^\gamma(f - Q_v f)\|_\infty \leq K_{v,\gamma} h^{4-|\gamma|} \max_{|\beta|=4} \left\| D^\beta f \right\|_\infty .$$

where $D^\beta = D^{\beta_1 \beta_2} = \frac{\partial^{|\beta|}}{\partial x^{\beta_1} \partial y^{\beta_2}}$, with $\beta_1 + \beta_2 = |\beta|$.

Example

$f(x, y) =$ Franke's function on $[0, 1]^2$



Example – *Approximation of the function*

G: uniform rectangular grid of 300×300 points in the domain

$$Ef = \max_{(u,v)\in G} |f(u,v) - Qf(u,v)|, \text{ for } Q = Q_1, Q_2$$

rf: numerical convergence order

	Q_1		Q_2	
$m_1 = m_2$	Ef	rf	Ef	rf
32	8.8(-4)		8.8(-4)	
64	6.0(-5)	3.9	6.0(-5)	3.9
128	3.9(-6)	4.0	3.9(-6)	4.0
256	2.4(-7)	4.0	2.4(-7)	4.0

Example – *Approximation of the gradient*

$$\nabla Ef = \max_{(u,v)\in G} \left(\left| \frac{\partial}{\partial x} f(u,v) - \frac{\partial}{\partial x} Qf(u,v) \right| + \left| \frac{\partial}{\partial y} f(u,v) - \frac{\partial}{\partial y} Qf(u,v) \right| \right),$$

for $Q = Q_1, Q_2$

∇rf: numerical convergence order

$m_1 = m_2$	Q_1 ∇Ef	Q_1 ∇rf	Q_2 ∇Ef	Q_2 ∇rf
32	8.9(-2)		4.5(-2)	
64	8.9(-3)	3.3	5.4(-3)	3.0
128	9.0(-4)	3.3	6.8(-4)	3.0
256	9.8(-5)	3.2	8.6(-5)	3.0

Example – *Approximation of the gradient*

G': grid of points of superconvergence

$$\nabla Ef = \max_{(u,v)\in G'} \left(\left| \frac{\partial}{\partial x} f(u,v) - \frac{\partial}{\partial x} Qf(u,v) \right| + \left| \frac{\partial}{\partial y} f(u,v) - \frac{\partial}{\partial y} Qf(u,v) \right| \right),$$

for $Q = Q_1, Q_2$

∇rf: numerical convergence order

	Q_1		Q_2	
$m_1 = m_2$	∇Ef	∇rf	∇Ef	∇rf
32	8.9(-2)		3.4(-2)	
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256	9.8(-5)	3.2	9.8(-6)	4.0

Comparison of the two methods

Near-best QI

- we impose the exactness on $\mathbb{P}_3(\mathbb{R}^2)$ and we minimize an upper bound for the QI infinity norm
- the construction of each functional is independent of the others

Comparison of the two methods

Near-best QI

- we impose the exactness on $\mathbb{P}_3(\mathbb{R}^2)$ and we minimize an upper bound for the QI infinity norm
- the construction of each functional is independent of the others

Superconvergent QI

- we impose the exactness on $\mathbb{P}_3(\mathbb{R}^2)$ and the interpolation condition for the gradient at the specific points for the monomials of $\mathbb{P}_4(\mathbb{R}^2) \backslash \mathbb{P}_3(\mathbb{R}^2)$ and, in case of free parameters, we minimize an upper bound for the QI infinity norm

  $$\Downarrow$$

  - more conditions and functionals involving more data points
  - loss of independence in the functional construction
- best performances in the numerical tests

Work in progress

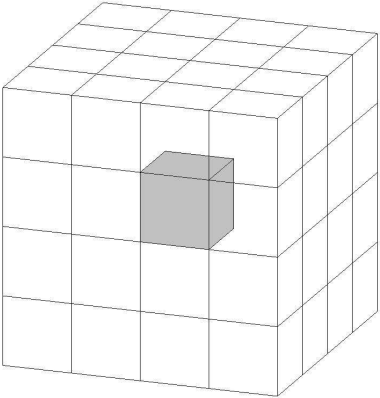
- Solution of integral equations on surfaces in $\mathbb{R}^3$ by spline quasi-interpolation

3D SPLINE SPACES

Partition of the domain $\Omega \subset \mathbb{R}^3$

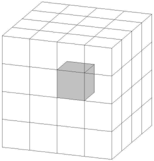
$$\Omega = [0, m_1 h] \times [0, m_2 h] \times [0, m_3 h] \subset \mathbb{R}^3$$

divided into equal cubes

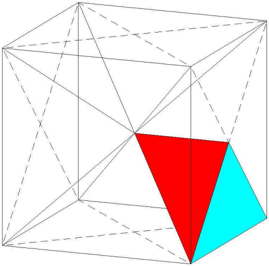


Trivariate spline space $S_4^2(\Omega, \mathcal{T}_{\mathbf{m}})$

Partition $\mathcal{T}_{\mathbf{m}}$, $\mathbf{m} = (m_1, m_2, m_3)$



subdivision of a cube
into 24 tetrahedra
(type-6 tetrahedral partition)



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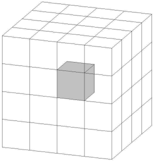

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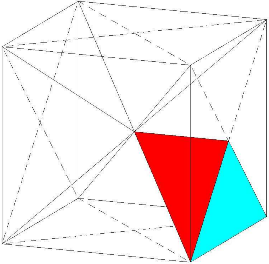
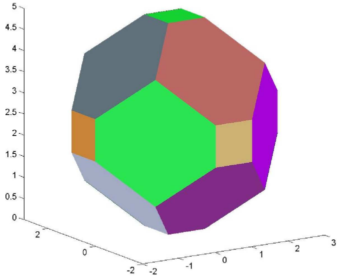


$\Downarrow$

Spline space $S_4^2(\Omega, \mathcal{T}_{\mathbf{m}})$

Trivariate spline space $S_4^2(\Omega, \mathcal{T}_{\mathbf{m}})$ [Peters, 1994]

Spline space generated by the scaled translates of the 7-direction box spline $B(x, y, z)$, whose supports overlap with Ω



Support of the 7-direction box spline $B(x, y, z)$:

Truncated rhombic dodecahedron contained in the cube $[-2, 3] \times [-2, 3] \times [0, 5]$ and centered at $\left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}\right)$

Optimal spline quasi-interpolants [Dagnino-Lamberti-Remogna, 2012-2014]

- Optimal spline quasi-interpolants exact on $\mathbb{P}_3(\mathbb{R}^3)$

$$Q : \mathcal{F} \to S_4^2(\Omega, \mathcal{T}_{\mathbf{m}})$$
$$f(x, y, z) \approx Qf(x, y, z)$$

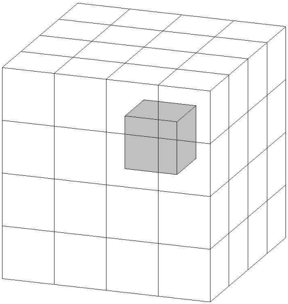
of near-best type, i.e. with coefficient functionals obtained by minimizing an upper bound for the QI infinity norm.

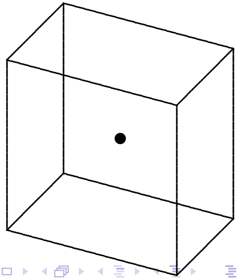
Quasi-interpolation nodes

$$\{M_{ijk} = (s_i, t_j, u_k)\}, \text{ with}$$

$$\begin{array}{llll} s_0 = 0, & s_i = (i - \frac{1}{2})h, \ 1 \le i \le m_1, & s_{m_1+1} = m_1 h \\ t_0 = 0, & t_j = (j - \frac{1}{2})h, \ 1 \le j \le m_2, & t_{m_2+1} = m_2 h \\ u_0 = 0, & u_k = (k - \frac{1}{2})h, \ 1 \le k \le m_3, & u_{m_3+1} = m_3 h. \end{array}$$

inside or on the boundary of Ω





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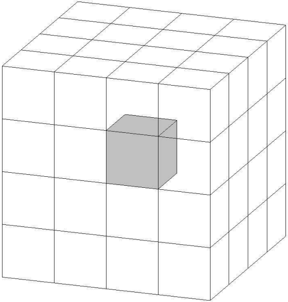
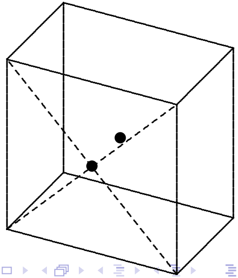
t_0 = 0, & t_j = (j - \frac{1}{2})h,\ 1 \le j \le m_2, & t_{m_2+1} = m_2 h \\
u_0 = 0, & u_k = (k - \frac{1}{2})h,\ 1 \le k \le m_3, & u_{m_3+1} = m_3 h.
\end{array}$$

inside or on the boundary of Ω





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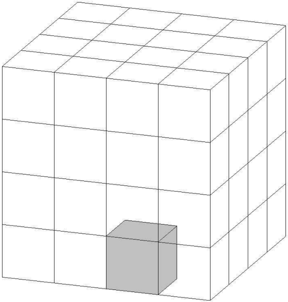
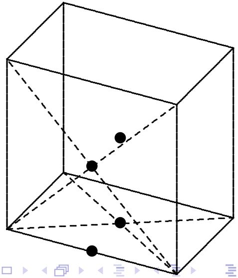
inside or on the boundary of Ω



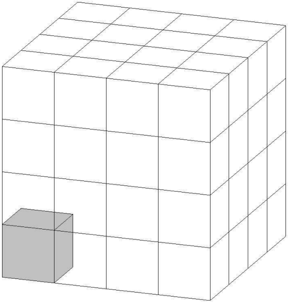


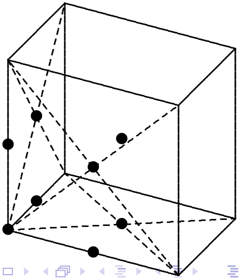
Quasi-interpolation nodes

$$\{M_{ijk} = (s_i, t_j, u_k)\}, \text{ with}$$

$$\begin{array}{lll} s_0 = 0, & s_i = (i - \frac{1}{2})h,\ 1 \le i \le m_1, & s_{m_1+1} = m_1 h \\ t_0 = 0, & t_j = (j - \frac{1}{2})h,\ 1 \le j \le m_2, & t_{m_2+1} = m_2 h \\ u_0 = 0, & u_k = (k - \frac{1}{2})h,\ 1 \le k \le m_3, & u_{m_3+1} = m_3 h. \end{array}$$

inside or on the boundary of Ω





Optimal spline quasi-interpolants [Dagnino-Lamberti-Remogna, 2012-2014]

- Optimal spline quasi-interpolants exact on $\mathbb{P}_3(\mathbb{R}^3)$

$$Q : \mathcal{F} \to S_4^2(\Omega, \mathcal{T}_{\mathbf{m}})$$
$$f(x,y,z) \approx Qf(x,y,z)$$

  of near-best type, i.e. with coefficient functionals obtained by minimizing an upper bound for the QI infinity norm.
- Quasi-interpolation nodes inside or on the boundary of Ω $\{M_{ijk} = (s_i, t_j, u_k)\}$, with

$$\begin{array}{lll} s_0 = 0, & s_i = (i - \frac{1}{2})h,\ 1 \le i \le m_1, & s_{m_1+1} = m_1 h \\ t_0 = 0, & t_j = (j - \frac{1}{2})h,\ 1 \le j \le m_2, & t_{m_2+1} = m_2 h \\ u_0 = 0, & u_k = (k - \frac{1}{2})h,\ 1 \le k \le m_3, & u_{m_3+1} = m_3 h. \end{array}$$

- Quasi-interpolation nodes also outside Ω [Remogna, 2011; Dagnino-Lamberti-Remogna, 2013]

Error estimates

Let $f \in C^r(\Omega)$, $r = 0, 1, 2, 3$. Then there exist constants $\bar{K}_r > 0$, such that

$$\|f - Qf\|_\infty \leq \bar{K}_r h^r \omega(D^r f, h).$$

If in addition $f \in C^4(\Omega)$ then there exists constant $\bar{K}_4 > 0$, such that

$$\|f - Qf\|_\infty \leq \bar{K}_4 h^4 \max_{|\beta|=4} \left\| D^\beta f \right\|_\infty .$$

Numerical tests

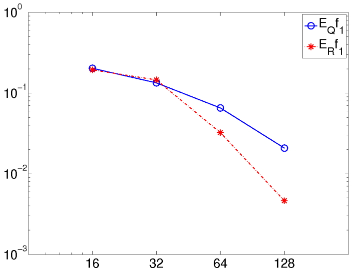
- Domain = $[a, b]^3$
- $h = \frac{b-a}{m}$, $m = m_1 = m_2 = m_3$, $m = 16, 32, 64, 128$
- G = 139 $\times$ 139 $\times$ 139 uniform grid of evaluation points in Ω
- $E_Q f = \max_{\mathbf{u}\in G} |f(\mathbf{u}) - Qf(\mathbf{u})|$, $Qf \in S_4^2(\Omega, \mathcal{T}_{\mathbf{m}})$
  $E_R f = \max_{\mathbf{u}\in G} |f(\mathbf{u}) - Rf(\mathbf{u})|$, $Rf \in S_{2,2}^1(\Omega, \mathcal{P}_{\mathbf{m}})$

  R is a spline QI in the space $S_{2,2}^1(\Omega, \mathcal{P}_{\mathbf{m}})$ of trivariate splines on prismatic partitions defined as tensor product of univariate and bivariate C^1 quadratic B-splines.
  R is obtained as blending sum of uni and bivariate C^1 quadratic spline QIs [Remogna-Sablonnière, 2011]
- $r_R f$, $r_Q f$ numerical convergence orders

The Marschner-Lobb function f_1

$$f_1(x,y,z) = \frac{1}{2(1+\beta_1)}\left(1 - \sin\frac{\pi z}{2} + \beta_1\left(1 + \cos\left(2\pi\beta_2\cos\left(\frac{\pi\sqrt{x^2+y^2}}{2}\right)\right)\right)\right)$$

with $\beta_1 = \frac{1}{4}$ and $\beta_2 = 6$
on the cube $[-1,1]^3$

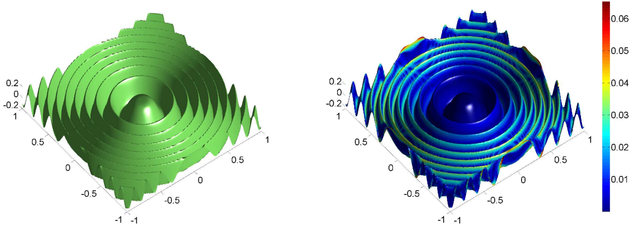


m	$E_Q f_1$	$r_Q f_1$	$E_R f_1$	$r_R f_1$
16	2.0(-1)		1.9(-1)	
32	1.3(-1)	0.6	1.5(-1)	0.4
64	6.5(-2)	1.0	3.2(-2)	2.2
128	2.1(-2)	1.7	4.6(-3)	2.8

The Marschner-Lobb function f_1

$$f_1(x,y,z) = \frac{1}{2(1+\beta_1)}\left(1-\sin\frac{\pi z}{2}+\beta_1\left(1+\cos\left(2\pi\beta_2\cos\left(\frac{\pi\sqrt{x^2+y^2}}{2}\right)\right)\right)\right)$$

with $\beta_1 = \frac{1}{4}$ and $\beta_2 = 6$ on the cube $[-1,1]^3$



(a) (b)

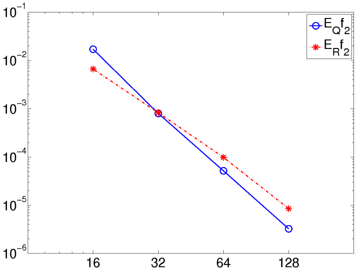
The isosurface obtained from (a) f_1 and (b) Qf_1, with $m = 64$, for the isovalue $\rho = 1/2$

The smooth trivariate test function of Franke type f_2

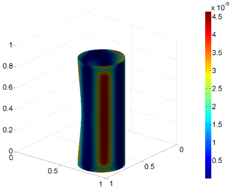
$$f_2(x,y,z) = \frac{1}{2}e^{-10((x-\frac{1}{4})^2+(y-\frac{1}{4})^2)} + \frac{3}{4}e^{-16((x-\frac{1}{2})^2+(y-\frac{1}{4})^2+(z-\frac{1}{4})^2)} + \frac{1}{2}e^{-10((x-\frac{3}{4})^2+(y-\frac{1}{8})^2+(z-\frac{1}{2})^2)} - \frac{1}{4}e^{-20((x-\frac{3}{4})^2+(y-\frac{3}{4})^2)}$$

on the cube $[0,1]^3$

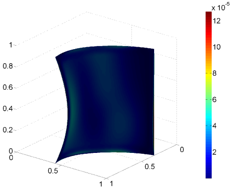
m	$E_Q f_2$	$r_Q f_2$	$E_R f_2$	$r_R f_2$
16	1.7(-2)		6.6(-3)	
32	8.0(-4)	4.4	8.2(-4)	3.0
64	5.2(-5)	3.9	9.8(-5)	3.1
128	3.3(-6)	4.0	8.5(-6)	3.5



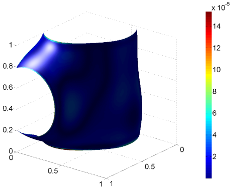
The smooth trivariate test function of Franke type f_2



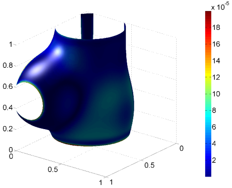
$\rho = -0.1$



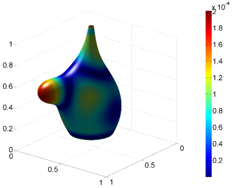
$\rho = 0$



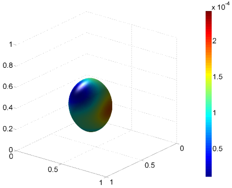
$\rho = 0.1$



$\rho = 0.2$



$\rho = 0.5$

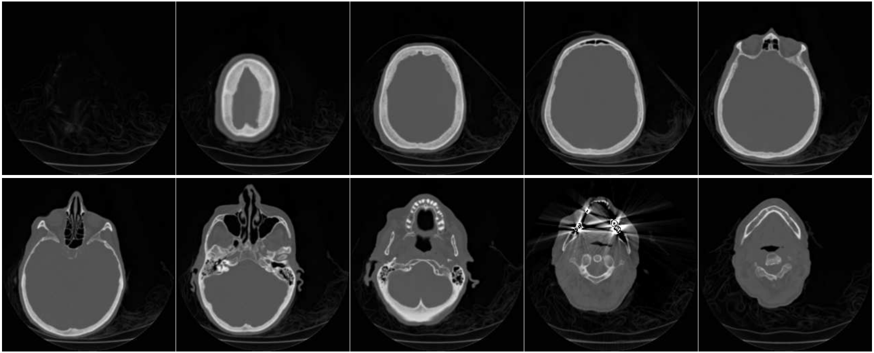


$\rho = 0.8$

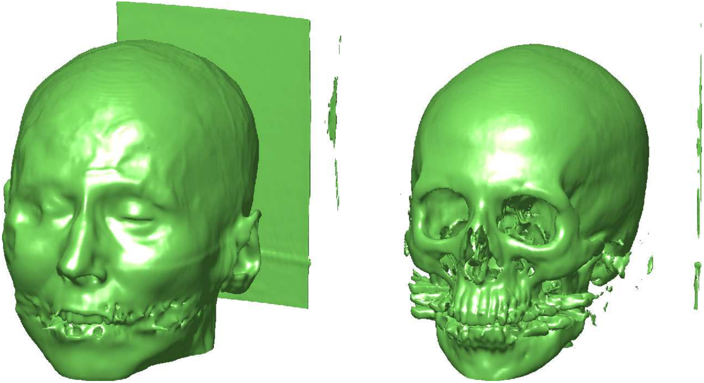
Isosurfaces of Qf_2 for $m = 32$, with different isovalues

Reconstruction of real world data – *CT Head data set*

Gridded volume data set consisting of $256 \times 256 \times 99$ data samples obtained from a CT scan of a cadaver head (courtesy of University of North Carolina)



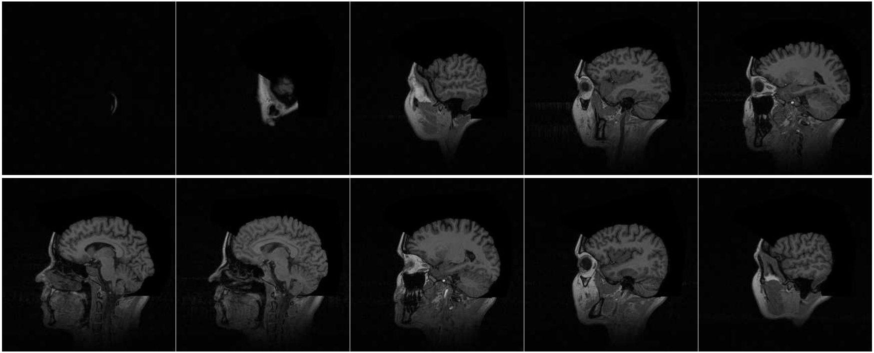
Reconstruction of real world data – *CT Head data set*



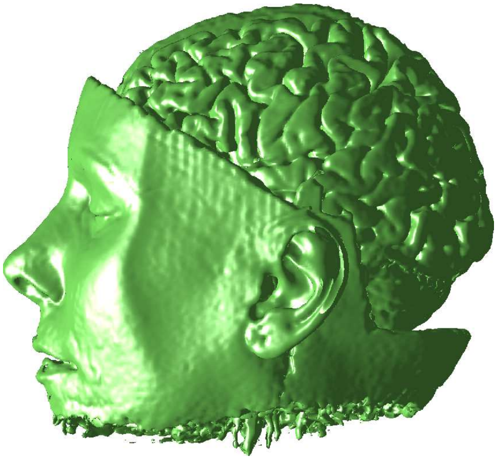
(a) (b)

Isosurfaces of the C^2 trivariate quartic spline approximating the *CT Head data set* with isovalues: (a) $\rho = 60$, (b) $\rho = 90$, with $\sharp G \approx 8.6 \cdot 10^6$ evaluation points

Gridded volume data set of $256 \times 256 \times 99$ data samples obtained from a MR study of head with skull partially removed to reveal brain (courtesy of University of North Carolina)



Reconstruction of real world data – *MR brain data set*



Isosurface of the C^2 trivariate quartic spline approximating the *MR brain data set* with isovalue $\rho = 40$, with $\sharp G \approx 8.6 \cdot 10^6$ evaluation points

Applications to numerical integration [Dagnino-Lamberti-Remogna, 2012-2013]

For any function $f \in C(\Omega)$, we consider the evaluation of the integral

$$I(f) = I(f;\Omega) := \int_{\Omega} f(x,y,z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

Applications to numerical integration [Dagnino-Lamberti-Remogna, 2012-2013]

For any function $f \in C(\Omega)$, we consider the evaluation of the integral

$$I(f) = I(f;\Omega) := \int_\Omega f(x,y,z) \; \mathrm{d}x \; \mathrm{d}y \; \mathrm{d}z,$$

by cubature rules defined by

$$I_Q(f) = I(Qf;\Omega) := \sum_{ijk} w_{ijk}^Q f(M_{ijk}),$$

with

- M_{ijk}: cubature nodes in Ω. They coincide with the quasi-interpolation nodes
- w_{ijk}^Q: cubature weights, linear combinations of $\int_{\Omega \cap suppB_{ijk}} B_{ijk}$
- the precision degree is 3, because Q is exact on $\mathbb{P}_3(\mathbb{R}^3)$
- if $f \in C^4(\Omega)$, then $\mid I(f) - I_Q(f) \mid = O(h^4)$

Example

- integration domain: $\Omega = [0,1]^3$
- $m_1 = m_2 = m_3 = m$, $h = 1/m$ and $m = 16, 32, 64, 128$
- integrand functions
  - $f_1(x,y,z) = e^{\left((x-0.5)^2+(y-0.5)^2+(z-0.5)^2\right)}$ (smooth test function), $I(f_1) = 0.7852115962$
  - $f_2 = \frac{27}{8}\sqrt{1-|2x-1|}\sqrt{1-|2y-1|}\sqrt{1-|2z-1|}$ (continuous test function), $I(f_2) = 1$

m	$\lvert I(f_1) - I_Q(f_1)\rvert$	rf_1	$\lvert I(f_2) - I_Q(f_2)\rvert$	rf_2
16	2.9(-5)		4.9(-3)	
32	1.9(-6)	3.9	2.4(-3)	1.1
64	1.3(-7)	3.9	9.3(-4)	1.3
128	8.1(-9)	4.0	3.5(-4)	1.4

Work in progress

- Systematic method for the construction of families of near-best C^2 quartic spline QIs on type-6 tetrahedral partitions of the space

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Thank you!