Generalized spline spaces and their optimal basis

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Outline

1. Polynomial vs. generalized splines
2. An approach for the construction of the“optimal" basis
3. Computation with generalized splines
4. Existence of the optimal basis

Polynomial vs. generalized splines [I]

- Let $[a, b]$ be a finite and closed interval and let

$$\boldsymbol{\Delta} = \{x_j\}_1^k \qquad \text{with} \qquad a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$$

be a partition of it into $k+1$ subintervals

$$I_j = [x_j, x_{j+1}), \quad j = 0, 1, \ldots, k-1, \quad I_k = [x_k, x_{k+1}].$$

- Let m be a positive integer and let $M = (m_1, \ldots, m_k)$ be a vector of integers with $1 \leqslant m_j \leqslant m$, $j = 1, 2, \ldots, k$.

Polynomial splines

$$S(\mathcal{P}_m, \boldsymbol{M}, \boldsymbol{\Delta}) := \Big\{ s \,\Big|\, \text{there exist polyomials } s_0, \ldots, s_k \text{ in } \mathcal{P}_m \text{ such that:}$$
$$\text{i) } s(x) = s_j(x) \text{ for } x \in I_j,\ j = 0, 1, \ldots, k$$
$$\text{ii) } D^r s_{j-1}(x_j) = D^r s_j(x_j),\ r = 0, \ldots, m - m_j - 1,\ j = 1, \ldots, k \Big\}$$

- The space $S(\mathcal{P}_m, \boldsymbol{M}, \boldsymbol{\Delta})$ has dimension $m + \sum_{j=1}^k m_j$

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  - $\langle 1, \cosh(a_j x), \sinh(a_j x), \cos(b_j x), \sin(b_j x) \rangle$, $a_j, b_j \in \mathbb{R}$, $a_j, b_j > 0$

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- exact reproduction of salient functions
- shape-preserving approximation
- alternative to the rational model (Non-Uniform Rational B-splines)

[Schoenberg; J. Math. Mech. 1964],[Jerome,Schumaker; J. Approx. Theory 1976],[Lyche,Winter; J. Approx. Theory 1979],[Barry; Constr. Approx. 1996],[Carnicer,Mainar,Peña; Constr. Approx. 2003],[Costantini,Lyche,Manni; Numer. Math. 2005],[Buchwald, Mühlbach; JCAM 2003],[Wang,Fang; JCAM 2008],[Ayalon,Dyn,Levin; J. Approx. Theory 2009],[Bosner,Rogina;Adv. Comput. Math. 2013],. . .

Suitable function spaces

- Applications in
  - Design, Geometric Modeling
    [Kvasov, Sattayatham; JCAM 1999],[Costantini; Comput. Aided Geom. Design 2000],[Mainar, Peña, Sánchez-Reyes; Comput. Aided Geom. Design 2001],. . . ,[Brilleaud, Mazure; Comput. Math. Appl. 2012, Numer. Algorithms 2014]
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- suitable spaces must have the ONTP basis
  - **O**ptimal
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  - **T**otally
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(For polynomials/polynomial splines: ONTP basis = Bernstein/B-spline basis)

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- suitable spaces must have the ONTP basis { Optimal, Normalized, Totally, Positive }

  (For polynomials/polynomial splines: ONTP basis = Bernstein/B-spline basis)
- non-uniform knots, multiple knots, parametric continuity

ONTP Bases for generalized spline spaces

- Find explicit expressions for the basis functions and/or computational algorithms for their evaluation
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▶ General approaches for a single generalized polynomial space

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- In the spline setting, approaches for the construction and analysis of existence of the ONTP basis are available for particular spaces only

  [Bosner,Rogina; Numer. Algor. 2007],[Costantini; Comput. Aided Geom. Design 2000],[Wang,Fang; JCAM 2008],[Xu,Wang;J. Comput. Sci. Technol. 2007]

How the Berstein Basis was introduced in CAGD

Rabut, Comput. Aided-Design 2000; Farouki, Comput. Aided Geom. Des. 2012

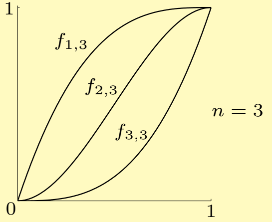
- Bézier basis for the space $\mathcal{P}_n$ of degree-n polynomials (1966):

$$f_{0,n} = 1, \quad f_{i,n}(t) = \sum_{j=i}^{n} (-1)^{i+j} \binom{n}{j} \binom{j-1}{i-1} t^j$$

with $i = 1, \ldots, n$ and $t \in [0,1]$

- Bézier curve: given an initial point $\boldsymbol{p}_0$ and n vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$

$$\boldsymbol{c}(t) = \boldsymbol{p}_0 + \sum_{i=1}^{n} \boldsymbol{a}_i f_{i,n}(t)$$



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- $f_{i,n}$ monotonically increasing

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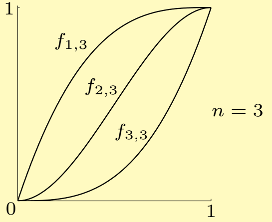

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- The Bernstein basis can be expressed in terms of the f's:

$B_{0,n} = 1 - f_{1,n}$ $\qquad$ $B_{i,n} = f_{i,n} - f_{i+1,n}$ for $i = 1, \ldots, n-1$ $\qquad$ $B_{n,n} = f_{n,n}$

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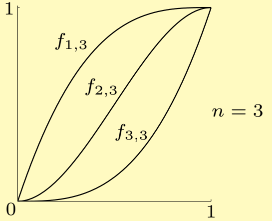

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- From the partition-of-unity property of the Bernstein basis: $f_{i,n} = \sum_{j=i}^{n} B_{j,n}$

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- $\boldsymbol{\Delta} = \{x_j\}$ knot partition, $\boldsymbol{M} = \{m_j\}$ multiplicities of $\{x_j\}$

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- $\boldsymbol{\Delta}^* = \{t_i\}$ extended partition

  $m_i^{\mathrm{R}} := \max\{p \geqslant 0 \mid t_i = t_{i+p}\} + 1$, $m_i^{\mathrm{L}} := \max\{p \geqslant 0 \mid t_{i-p} = t_i\} + 1$

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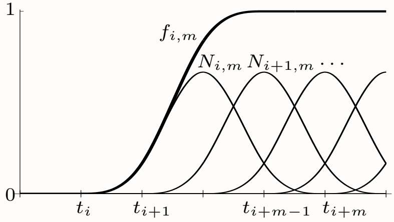

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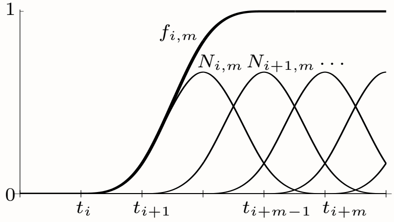

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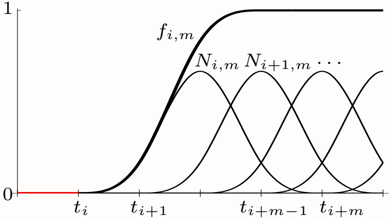

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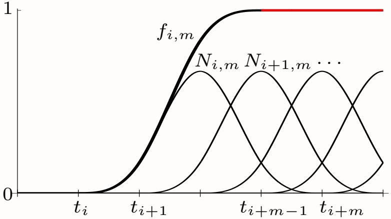

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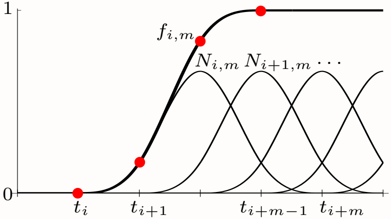

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  $f_{i,m}$ is $C^{m-m_{i+m-1}^L-1}$ at t_{i+m-1} ⇐ the least continuity among $\{N_{j,m}\}$

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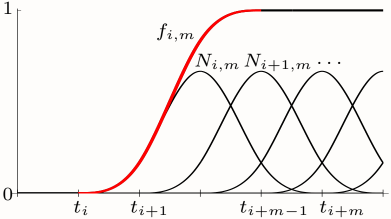

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  $f_{i,m}$ is C^{m-m_j-1} at x_j, for $t_i < x_j < t_{i+m-1}$

  (iv) $f_{i,m}$ is monotonically increasing ⇐ from the zero property of polynomial splines, $f'_{i,m} > 0$

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Transition function

We name *transition function* any function that satisfies (i)–(ii)–(iii)

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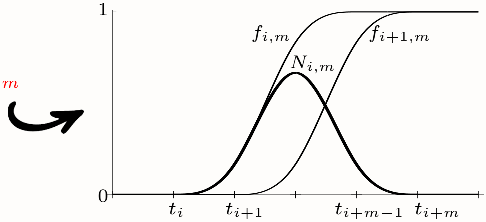

A construction for the polynomial B-spline basis [II]

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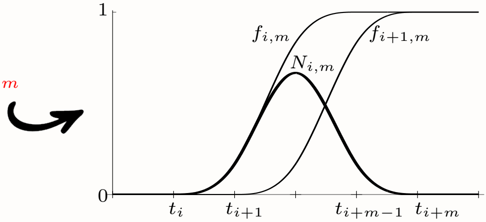

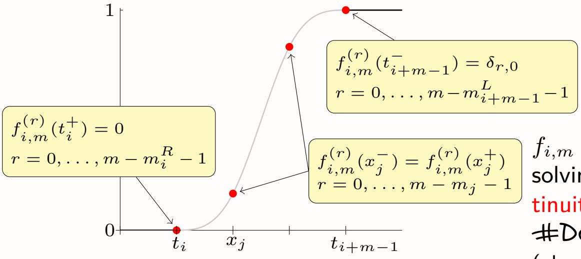

$f_{i,m}$ can be uniquely determined by solving the linear system of the continuity conditions at the knots:

#DoF=#CC

(also in the case of multiple knots)

Generalizing the construction

- $\mathbf{\Delta} = \{x_j\}$ knot partition, $\boldsymbol{M} = \{m_j\}$ multiplicities of $\{x_j\}$, $\mathbf{\Delta}^* = \{t_i\}$ extended partition
- We let each piece of $f_{i,m}$ belong to a different m-dimensional function space $\mathcal{U}_{j,m} = \langle 1, u_{j,2}, \ldots, u_{j,m} \rangle$ associated with $[x_j, x_{j+1}]$

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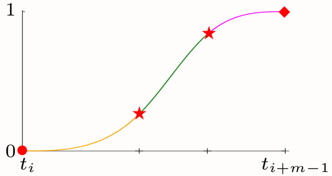

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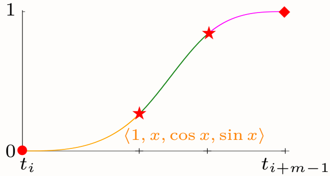

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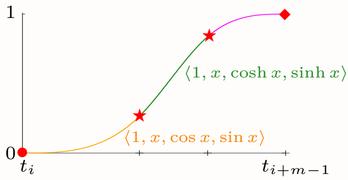

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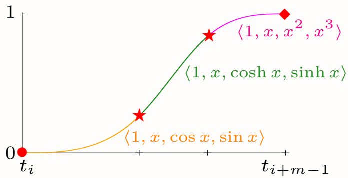

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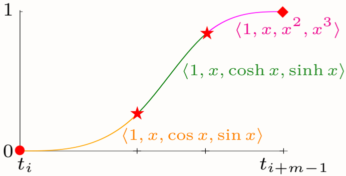
● $f_{i,m}^{(r)}(t_i^+) = 0 \qquad r = 0, \ldots, m - m_i^R - 1$

★ $f_{i,m}^{(r)}(x_j^-) = f_{i,m}^{(r)}(x_j^+) \qquad r = 0, \ldots, m - m_j - 1$

$\forall j$ s.t. $t_i < x_j < t_{i+m-1}$

◆ $f_{i,m}^{(r)}(t_{i+m-1}^-) = \delta_{r,0} \qquad r = 0, \ldots, m - m_{i+m-1}^L - 1$



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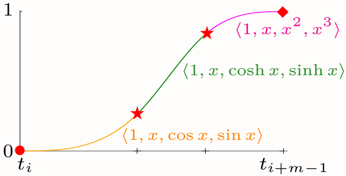

- $\exists!$ solution by properly choosing $\mathcal{U}_{j,m}$ ➠ QEC spaces

EC and QEC spaces

Quasi Extended Chebyshev space [Mazure, Numer. Math. 2008]

An m-dimensional space $\mathcal{U} \subset C^{m-2}(I)$, $m \geqslant 2$, is QEC on $I \subset \mathbb{R}$ if:

- any Hermite interpolation problem in m data in I, with at least two distinct points, has a unique solution in $\mathcal{U}$
- equivalently, for $m > 2$, any nonzero element of $\mathcal{U}$ with at least two distinct zeros vanishes at most $m-1$ times in I, counting multiplicities

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- $\mathcal{U}$ QEC $\Longrightarrow$ $\int \mathcal{U}$ is QEC, $g(x) \cdot \mathcal{U}$ is QEC for any $g(x) > 0$, $D\mathcal{U}$ is not QEC in general
- $\mathcal{U}_m = \langle 1, u_2, \ldots, u_m \rangle$ and $D\mathcal{U}_m = \langle u_2', \ldots, u_m' \rangle$ are QEC spaces on I $\Longleftrightarrow$ $\mathcal{U}_m$ has the Bernstein (ONTP) basis on I

Generalized spline spaces and ONTP bases

- $\boldsymbol{\Delta} = \{x_j\}$ knot partition, $\boldsymbol{M} = \{m_j\}$ multiplicities of $\{x_j\}$, $\boldsymbol{\Delta}^* = \{t_i\}$ extended partition
- $\boldsymbol{\mathcal{U}}_m := \{\mathcal{U}_{j,m}\}$ such that $\mathcal{U}_{j,m} = \langle 1, u_{j,2}, \dots, u_{j,m} \rangle$ and $D\mathcal{U}_{j,m} = \langle u'_{j,2}, \dots, u'_{j,m} \rangle$ are QEC on $[x_j, x_{j+1}]$, $\forall j$

Piecewise quasi Chebyshevian spline space

$$S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta}) := \Big\{ s \,\Big|\, \text{there exist } s_j \in \mathcal{U}_{j,m}, \forall j, \text{ such that:} \\ \text{i) } s(x) = s_j(x) \text{ for } x \in [x_j, x_{j+1}), \forall j; \\ \text{ii) } D^r s_{j-1}(x_j) = D^r s_j(x_j) \text{ for } r = 0, \dots, m - m_j - 1, \forall j \Big\}$$

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- $\{f_{i,m}\}$ is a basis for $S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta})$
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- $f_{i,m}$ vanishes *at least* $m - m_i^R$ times at t_i (exactly if $m_i^R > 1$) and $1 - f_{i,m}$ vanishes *at least* $m_{i+m-1}^L - 1$ times at t_{i+m-1} (exactly if $m_{i+m-1}^L > 1$)

B-spline and Bernstein Bases

B-spline basis

We say that $N_{i,m}$, $i = 1, \ldots, m + K$ is the B-spline basis of $S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta})$ if

i) *support property*: $N_{i,m}(x) = 0$, $x \notin (t_i, t_{i+m})$

ii) *positivity property*: $N_{i,m}(x) > 0$, $x \in (t_i, t_{i+m})$

iii) *partition of unity property*: $\sum_i N_{i,m}(x) = 1$, $\forall x \in [a, b]$

iv) *endpoint property*: $N_{i,m}$ vanishes $m - m_i^{\mathrm{R}}$ times at t_i (exactly if $m_i^r > 1$) and $m - m_{i+m}^{\mathrm{L}}$ times at t_{i+m} (exactly if $m_{i+m}^L > 1$)

Bernstein basis

Let $\mathcal{U}_m \subset C^{m-2}(I)$ be an m-dimensional space. Given $a, b \in I$, $a < b$, we say that $B_{i,m}$, $i = 0, \ldots, m - 1$ is the Bernstein basis of $\mathcal{U}_m$ relative to $[a, b]$ if

i) *zero property*:

- $B_{0,m}(a) \neq 0$ and $B_{0,m}$ vanishes $m - 1$ times at b
- $B_{m-1,m}(b) \neq 0$ and $B_{m-1,m}$ vanishes $m - 1$ times at a
- for $1 \leqslant i \leqslant m - 2$, $B_{i,m}$ vanishes exactly i times at a and exactly $m - 1 - i$ times at b

ii/iii) *positivity/normalization properties*: $B_{i,m}(x) > 0$, $x \in (a, b)$, $\sum_i B_{i,m}(x) = 1$

Computing with generalized splines [1]

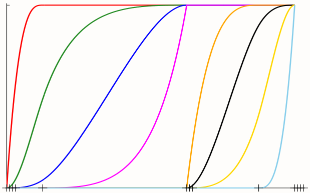
- Many approaches to address specific spaces
  - GB-splines [Kvasov, Sattayatham, JCAM 1999]
  - geometric construction [Costantini, CAGD 2000; Costantini, Manni, Rend. Mat. 2006]
  - generalized divided differences [Muhlbach, JCAM 2006]
  - integral recurrence relation [Bister, Prautzsch, *Curv. and Surf. in CAGD* 1997]

- Difficult to apply for spline spaces of very general form, i.e.
  - non-uniform and multiple knots
  - $\mathcal{U}_{j,m}$ different on each interval $[x_j, x_{j+1})$
  - spaces $\mathcal{U}_{j,m}$ have high dimension ($m > 4$)
  - many of the generators of $\mathcal{U}_{j,m} = \langle 1, u_{j,2}, \dots, u_{j,m} \rangle$ are non-polynomial functions

Computing with generalized splines [II]

- Transition functions easily allow for handling any kind of space and mixing different spaces

$\{f_{i,m}\}$



$\{N_{i,m}\}$

$m = 4$

$\mathcal{U}_{1,m} = \mathcal{U}_{4,m} = \langle 1, x, x^2, x^3 \rangle \qquad \mathcal{U}_{2,m} = \mathcal{U}_{3,m} = \langle 1, x, \cosh x, \sinh x \rangle$

$[a, b] = [0, 8] \qquad \boldsymbol{\Delta} = \{1, 5, 7\} \qquad \boldsymbol{M} = \{1, 3, 1\}$

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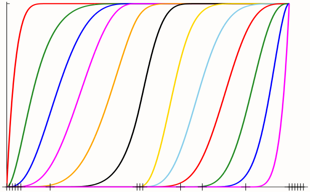

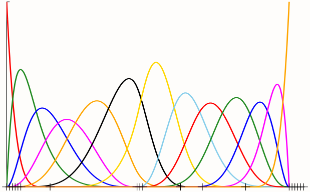

$\{N_{i,m}\}$

$m = 6$

$\mathcal{U}_{j,m} = \langle 1, x, \cos x, \sin x, \cosh x, \sinh x \rangle \ \forall j$

$[a, b] = [0, 6.5] \qquad \boldsymbol{\Delta} = \{1, 3, 4, 4.5, 5.5\} \qquad \boldsymbol{M} = \{1, 3, 1, 1, 1\}$

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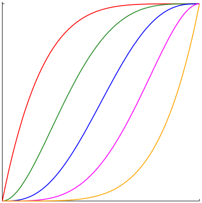

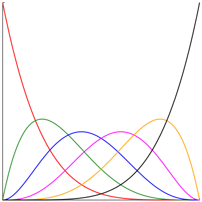

$\{B_{i,m}\}$

$m = 6$

$\mathcal{U}_m = \langle 1, x, \cos x, \sin x, \cosh(\phi x), \sinh(\phi x) \rangle,\ \phi = 1, \ldots, 20$

$[a, b] = [0, 1] \qquad \boldsymbol{\Delta} = \emptyset \qquad \boldsymbol{M} = \emptyset$

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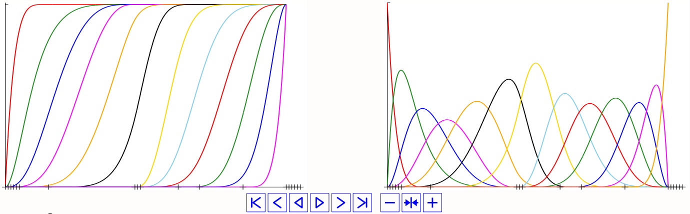

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Some insights on existence

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  - Bernstein basis: existence is characterized in terms of critical length of the QEC space [Carnicer, Mainar, Peña, Mazure, Brilleaud]

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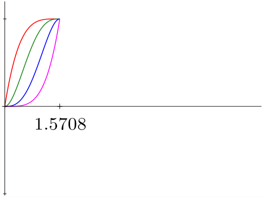
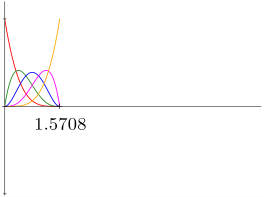


Bernstein basis:



$\{B_{i,m}\}$

$m = 5$

$\mathcal{U}_m = \langle 1, \cosh x \cos x, \cosh x \sin x, \sinh x \cos x, \sinh x \sin x \rangle$

$[a,b] = [0,d]$, $d = \frac{\pi}{2}, \ldots, 2\pi$ $\quad \boldsymbol{\Delta} = \emptyset \quad \boldsymbol{M} = \emptyset$

Brilleaud and Mazure, Comput. Math. Appl. 2012: $\ell \in \left(\pi, \frac{3}{2}\pi\right)$

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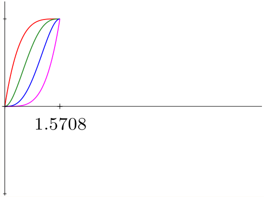

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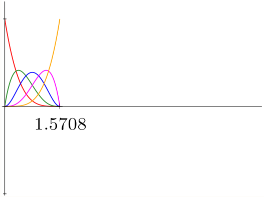

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$\mathcal{U}_m = \langle 1, \cosh x \cos(\theta x), \cosh x \sin(\theta x), \sinh x \cos(\theta x), \sinh x \sin(\theta x) \rangle$

$[a, b] = [0, d]$, $d = \frac{\pi}{2}, \ldots, 2\pi$ $\quad \boldsymbol{\Delta} = \emptyset$ $\quad \boldsymbol{M} = \emptyset$

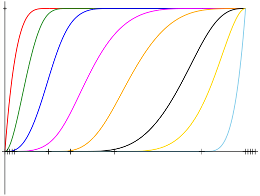
Brilleaud and Mazure, Comput. Math. Appl. 2012: $\ell \in \left(\frac{\pi}{\theta}, \frac{3}{2} \frac{\pi}{\theta} \right)$

Some insights on existence

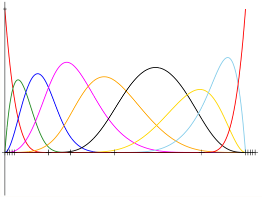
- For generalized spline spaces, an "optimal" basis does not always exist
  - Bernstein basis: existence is characterized in terms of critical length of the QEC space [Carnicer, Mainar, Peña, Mazure, Brilleaud]

- What about transition functions?

$\{f_{i,m}\}$



B-spline basis:



$\{N_{i,m}\}$

$m = 5$

$\theta_j = 1$ except $\theta_3 = 1, \ldots, 3$

$\mathcal{U}_{j,m} = \langle 1, \cosh x \cos(\theta_j x), \cosh x \sin(\theta_j x), \sinh x \cos(\theta_j x), \sinh x \sin(\theta_j x) \rangle,$

$[a, b] = [0, 5.5]$ $\quad \boldsymbol{\Delta} = \{1, 1.5, 2.5, 4.5\}$ $\quad \boldsymbol{M} = \{1, 1, 1, 1\}$

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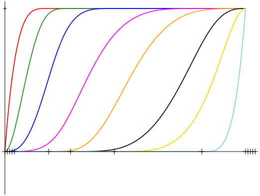

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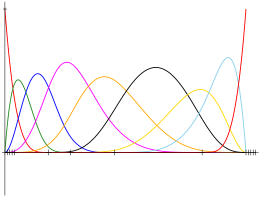

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- Existence of the B-spline basis depends on the length of each knot interval (particular cases studied by [Schumaker, Mazure, Brilleaud])

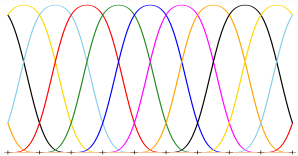
B-spline vs. ONTP Basis

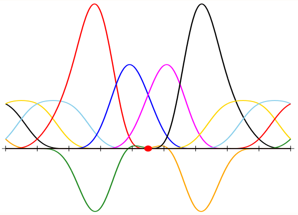
- A spline space $\hat{S}(\hat{\boldsymbol{\mathcal{U}}}_m, \hat{\boldsymbol{M}}, \hat{\boldsymbol{\Delta}})$ is obtained from $S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta})$ by knot insertion if
  - $S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta})$ and $\hat{S}(\hat{\boldsymbol{\mathcal{U}}}_m, \hat{\boldsymbol{M}}, \hat{\boldsymbol{\Delta}})$ have section spaces of the same dimension, that is m
  - $S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta}) \subset \hat{S}(\hat{\boldsymbol{\mathcal{U}}}_m, \hat{\boldsymbol{M}}, \hat{\boldsymbol{\Delta}})$

Equivalent conditions

i) $S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta})$ has the B-spline basis and any spline space obtained from it by knot insertion has the B-spline basis too

ii) There exists the Optimal Normalized Totally Positive basis in $S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta})$





$x_{j+1} - x_j = 3.5,\ m_j = 1,\ \mathcal{U}_{j,m} = \langle 1, x, x^2, \cos x, \sin x \rangle,\ \forall j$

ONTP Basis and weight functions

Weight functions

A sequence $(w_0, \ldots, w_{m-2})$ is a system of piecewise weight functions for $S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta})$ if

i) $w_k > 0$, $\forall k = 0, \ldots, m-2$

ii) w_k is $C^{m-k-m_{j,k}-1}$ at x_j, with $m_{j,k} := \min(m_j, m-k)$

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- With any system of p.w.f., we can associate piecewise generalized derivatives

$$L_0 v = \frac{v}{w_0}, \qquad L_k v = \frac{1}{w_k} D L_{k-1} v, \quad k = 1, \ldots, m-2$$

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Theorem [Antonelli, B., Casciola, Romani; 2014]

A generalized spline space $S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta})$ containing constants has the ONTP basis $\Longleftrightarrow$ there exists a system of p.w.f. and a spline space $V := S(L_{m-2}\, \boldsymbol{\mathcal{U}}_m, \boldsymbol{M}_{m-2}, \boldsymbol{\Delta})$ of dimension $2 + \sum_j m_{j,m-2}$, s.t. $1 \in V$ and

$$S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta}) = \{ s \text{ is } C^{m-m_j-1} \text{ at } x_j \mid L_{m-2} s \in V \}$$

ONTP Basis and weight functions

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Theorem [Antonelli, B., Casciola, Romani; 2014]

A generalized spline space $S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta})$ containing constants has the ONTP basis $\Longleftrightarrow$ there exists a system of p.w.f. and a spline space $V := S(L_{m-2}\,\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}_{m-2}, \boldsymbol{\Delta})$ of dimension $2 + \sum_j m_{j,m-2}$, s.t. $1 \in V$ and

$$S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta}) = \{s \text{ is } C^{m-m_j-1} \text{ at } x_j \mid L_{m-2} s \in V\}$$

- $w_k \in S(DL_{k-1}\,\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}_{k-1}, \boldsymbol{\Delta})$, $\boldsymbol{M}_{k-1} = \{m_{j,k-1}\}$
- The space $S(L_k\,\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}_k, \boldsymbol{\Delta})$ has dimension $(m-k) + \sum_j m_{j,k}$,

Existence by means of transition functions

- The transition functions allow us to generate a particular sequence of p.w.f. for $S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta})$

$$\begin{aligned} w_0 &= 1 \\ w_{k+1} &= \sum_i Df_{i,m-k}, \qquad k = 0, \ldots, m-3 \\ f_{i,m-k-1} &= \frac{\sum_{\ell \geqslant i} Df_{\ell,m-k}}{w_{k+1}} \end{aligned}$$

where $\quad f_{i,m} \in S(\boldsymbol{\mathcal{U}}_m, \boldsymbol{M}, \boldsymbol{\Delta}) \quad$ and $\quad f_{i,m-k} \in S(L_k\, \boldsymbol{\mathcal{U}}_m, \boldsymbol{M}_k, \boldsymbol{\Delta})$

- By construction $w_k \in C^{m-k-m_{j,k}-1}$ at x_j

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- By construction $w_k \in C^{m-k-m_{j,k}-1}$ at x_j

Theorem

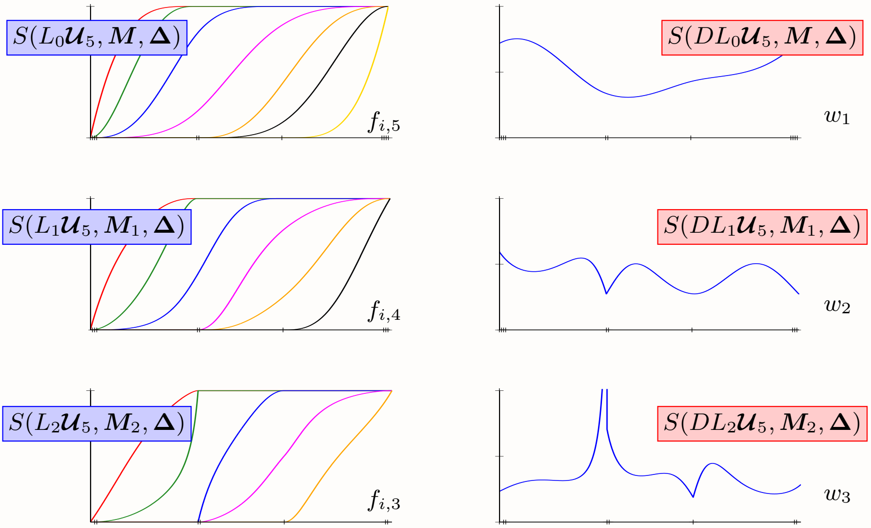
The transition functions $\{f_{i,m-k}\}$ are positive and monotonically increasing in all the spaces $S(L_k\, \boldsymbol{\mathcal{U}}_m, \boldsymbol{M}_k, \boldsymbol{\Delta})$, $k = 0, \ldots, m-3 \iff w_{k+1} > 0$, $\forall k = 0, \ldots, m-3$

Consequence:

- all $f_{i,m-k}$ are positive and monotonically increasing $\Longrightarrow$ ONTP
- one of the $f_{i,m-k}$ is not positive/monotonically increasing $\Longrightarrow$ ~~ONTP~~

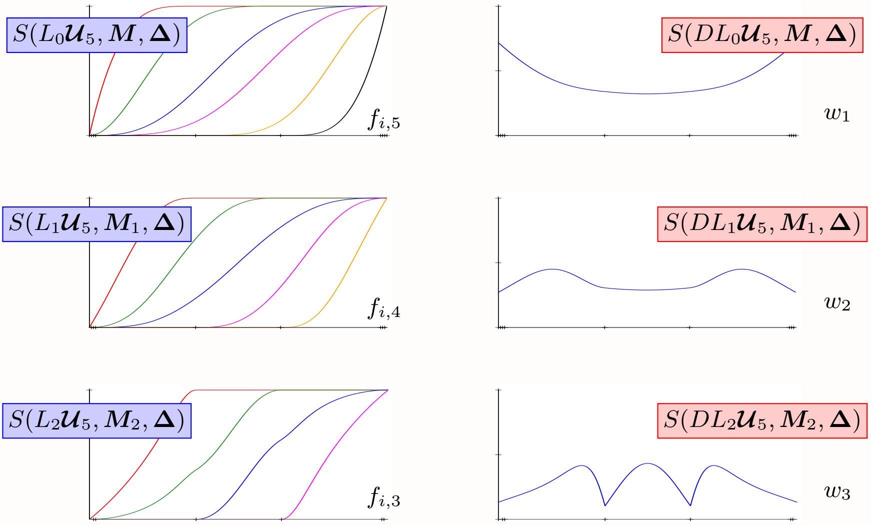
Example [1]

$S(\boldsymbol{\mathcal{U}}_5, \boldsymbol{M}, \boldsymbol{\Delta})$, $\mathcal{U}_{j,5} = \langle 1, x, x^2, \cos x, \sin x \rangle$, $[a,b] = [0,7]$, $\boldsymbol{\Delta} = \{2.5, 4.5\}$, $\boldsymbol{M} = \{2, 1\}$



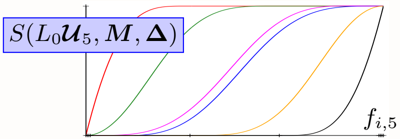
Example [II]

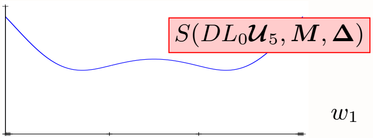
$S(\boldsymbol{\mathcal{U}}_5, \boldsymbol{M}, \boldsymbol{\Delta})$, $\mathcal{U}_{j,5} = \langle 1, x, x^2, \cos x, \sin x \rangle$, $[a,b] = [0,7]$, $\boldsymbol{\Delta} = \{2.5, 4.5\}$, $\boldsymbol{M} = \{1,1\}$

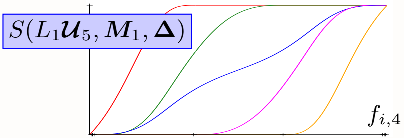


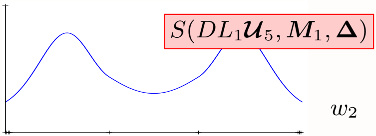
Example [III]

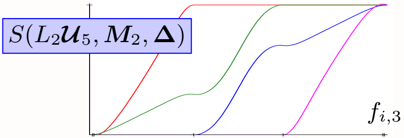
$S(\boldsymbol{\mathcal{U}}_5, \boldsymbol{M}, \boldsymbol{\Delta})$, $\mathcal{U}_{j,5} = \langle 1, x, x^2, \cos x, \sin x \rangle$, $[a,b] = [0,10]$, $\boldsymbol{\Delta} = \{3.5, 6.5\}$, $\boldsymbol{M} = \{1,1\}$

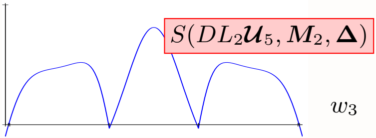












Thank you!