Fekete Points on Curves, Hankel Determinants and the Moment Problem

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Fekete Points

- $K \subset \mathbb{C}^d$ a compact set
- $\mathcal{B}_n = \{P_1, P_2, \cdots, P_N\}$ a Basis for the polynomials of degree $\leq n$, restricted to K
- $z_1, z_2, \cdots, z_N \in K$
- $[P_i(z_j)] \in \mathbb{C}^{N\times N}$ is the Vandermonde matrix
- Fekete points, $f_1, f_2, \cdots, f_N \in K$, maximize the determinant of the Vandermonde matrix, $|\mathrm{vdm}(z_1, \cdots, z_N)|$
- Transfinite Diameter is $\lim_{n\to\infty} |\mathrm{vdm}(f_1, \cdots, f_N)|^{1/\ell_n}$

Quadratic Curves

- Consider quadratic curve in $\mathbb{C}^2$, with coefficient of y^2 not equal to zero
- Basis $\mathcal{B}_n = \{1, x, x^2, \cdots, x^n\} \cup y\{1, x, x^2, \cdots, x^{n-1}\}$, $N = 2n+1$
- Set $K = \{(z, A(z)) \,:\, |z| \le 1\}$ with ....
- .... $A(z) = \sum_{j=0}^{\infty} a_j z_j$, radius of convergence > 1
- Points $X_n := \{(\omega^k, A(\omega^k)) \,:\, 0 \le k \le 2n\}$, $\omega^{2n+1} = 1$

A simplification

- Replace $A(z)$ by its polynomial interpolant $Q_{2n}(z)$ at X_n
- Points $X_n = \{(\omega^k, Q_{2n}(\omega^k)) \,:\, 0 \le k \le 2n\}$
- Basis element yx^r can be replaced by $x^r Q_{2n}(x)$
- Question: What is $x^r Q_{2n}(x)$ at ω^s?

Simplest Case

- Evaluate $xQ_{2n}(x)$ at $x = \omega$
- $$\begin{aligned} \omega Q_{2n}(\omega) &= \omega\big(q_0 + q_1\omega + q_2\omega^2 \cdots + q_{2n}\omega^{2n}\big) \\ &= \qquad q_0\omega + q_1\omega^2 + \cdots + q_{2n-1}\omega^{2n} + q_{2n}\omega^{2n+1} \\ &= q_{2n} + q_0\omega + q_1\omega^2 + \cdots + q_{2n-1}\omega^{2n} \\ &= Q_{2n}^{(1)}(\omega) \end{aligned}$$
- where $Q_{2n}^{(1)}(x)$ is the polynomial obtained by cyclically shifting the coefficients one to the right

In General

- $x^r Q_{2n}(x)$ at $x = \omega^s$ equals $Q_{2n}^{(r)}(\omega^s)$
- where $Q_{2n}^{(r)}(x)$ obtained by cyclically shifting by r to the right
- Replace the basis by

$$\widetilde{\mathcal{B}}_n := \{1, x, x^2, \cdots, x^n, Q_{2n}^{(0)}(x), Q_{2n}^{(1)}(x), \cdots, Q_{2n}^{(n-1)}(x)\}$$

- Point set becomes univariate:

$$\widetilde{X}_n := \{1, \omega, \omega^2, \cdots, \omega^{2n}\}$$

Transition to Standard Basis

- $$\begin{bmatrix} 1 \\ x \\ \cdot \\ \cdot \\ x^n \\ Q_{2n}^{(0)}(x) \\ Q_{2n}^{(1)}(x) \\ \cdot \\ \cdot \\ Q_{2n}^{(n-1)}(x) \end{bmatrix} = T \begin{bmatrix} 1 \\ x \\ \cdot \\ \cdot \\ x^n \\ x^{n+1} \\ x^{n+2} \\ \cdot \\ \cdot \\ x^{2n} \end{bmatrix}$$

The Transition Matrix T

$$T = \begin{bmatrix} I_{n+1} & & 0 & & & \\ & & & & & \\ & q_{n+1} & q_{n+2} & \cdot & \cdot & q_{2n} \\ & q_n & q_{n+1} & \cdot & \cdot & q_{2n-1} \\ * & \cdot & & & & \cdot \\ & \cdot & & & & \cdot \\ & q_2 & q_3 & \cdot & \cdot & q_{n+1} \end{bmatrix}$$

The Determinant of T

$$\det(T) = \begin{vmatrix} q_{n+1} & q_{n+2} & \cdot & \cdot & q_{2n} \\ q_n & q_{n+1} & \cdot & \cdot & q_{2n-1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ q_2 & q_3 & \cdot & \cdot & q_{n+1} \end{vmatrix} \qquad \text{(Toeplitz Form)}$$

$$= \pm \begin{vmatrix} q_2 & q_3 & \cdot & \cdot & q_{n+1} \\ q_3 & q_4 & \cdot & \cdot & q_{n+2} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ q_{n+1} & q_{n+2} & \cdot & \cdot & q_{2n} \end{vmatrix} \qquad \text{(Hankel Form)}$$

The Vandermonde Determinant

The Vandermonde Determinant becomes

$$\mathrm{vdm}(\widetilde{X}_n; \widetilde{\mathcal{B}}_n) = \det(T) \times \mathrm{vdm}(\widetilde{X}_n; \{1, x, x^2, \cdots, x^{2n})\}$$

A classical Vandermonde Determinant

The Interpolant at the *m*th roots of unity

Lemma

The interpolant of $F(z)=\sum_{j=0}^\infty a_j z^j$ *at the* m m*th roots of unity is given by*

$$P_{m-1}(z)=\sum_{k=0}^{m-1} q_k z^k$$

where

$$q_k=\sum_{j=0}^\infty a_{k+jm}.$$

The Generating Function

Definition

The (formal) power series $F(z) = \sum_{k=1}^{\infty} a_k z^k$ is said to be the (shifted) generating function of the Hankel matrices $H_n \in \mathbb{R}^{n\times n}$ if

$$(H_n)_{i,j} = a_{i+j-1}.$$

- Note that the summation begins at $k = 1$!!

A First Example

- Consider $F(z)=\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}z^k=-\log(1-z)\ \ (|z|<1)$
- $H_n=[1/(i+j-1)]$, the classical Hilbert matrix
- $\det(H_n)=\dfrac{\left(\prod_{j=1}^{n-1}j!\right)^4}{\prod_{j=1}^{2n}j!}$
- $\displaystyle\lim_{n\to\infty}(\det(H_n))^{1/n^2}=\frac{1}{4}$
- which is the transfinite diameter of the interval $[0,1]$

Gram Matrices

- Since $H_{i,j} = \dfrac{1}{i+j-1} = \int_0^1 x^{i-1}x^{j-1}dx$
- H_n is the Gram matrix for $\{1, x, x^2, \cdots, x^{n-1}\}$ for dx on $[0,1]$

Definition

A measure μ is in the Stahl-Totik class Reg if $\lim_{n\to\infty} \|P_n\|_{L_2(\mu)}^{1/n} = \mathrm{tfd}(K)$, where P_n is the monic orthogonal polynomial.

Theorem

(Stahl-Totik) The measure $\mu \in \mathrm{Reg}$ iff there is a Nikolski type inequality,

$$\|p\|_K \le c_n \|p\|_{L_2(\mu)}, \quad \lim_{n\to\infty} c_n^{1/n} = 1$$

for all polynomials with $\deg(p) \le n$.

Bloom's Theorem

Theorem

(Tom Bloom, 2014) Suppose that $G_n(\mu) \in \mathbb{R}^{n\times n}$ is the Gram matrix for the measure μ with

$$(G_n)_{i,j} = \int x^{i-1}x^{j-1}d\mu.$$

Then

$$\lim_{n\to\infty} (\det(G_n(\mu)))^{1/n^2} = \mathrm{tfd}(K) \iff \mu \in \mathrm{Reg}.$$

Further Results

For the Generating Function $F(z)$ let
$f(z) = F(1/z) = \sum_{k=1}^{\infty} a_k/z^k$

Theorem

(Polya-Goluzin Formula) Suppose that $f(z)$ is analytic outside *a compact set $E \subset \mathbb{C}$. Then*

$$n!\mathrm{det}(H_n) = \frac{1}{(2\pi i)^n}\int_\gamma \cdot\cdot \int_\gamma f(z_1)f(z_2)\cdots f(z_n)\mathrm{vdm}^2(z_1,\cdots,z_n)dz_1\cdots dz_n.$$

Theorem

(Polya, 1928) We have

$$\limsup_{n\to\infty} |\mathrm{det}(H_n)|^{1/n^2} \leq \mathrm{tfd}(K).$$

Pick Functions

Theorem

(Liu and Pego, 2014) The sequence $(c_k)_{0\le k<\infty}$ *is the moment sequence of a measure supported on* $[0,1]$ *iff*

$$F(z)=\sum_{k=1}^{\infty} c_{k-1}z^k$$

is a so-called Pick function analytic on $(-\infty,1)$.

Definition

$F(z)$ is a Pick function analytic on $(-\infty,1)$ if

- $F(z)$ is anayltic in the upper half-plane, and leaves it invariant.
- $F(z)$ takes real values on $(-\infty,1)$ and admits analytic continuation across this interval.

Recovery of the Measure

In the case that $a_k = \int x^{k-1} d\mu$, then

$$f(z) = \sum_{k=1}^{\infty} \left(\int x^{k-1} d\mu \right) z^{-k}$$
$$= -\int \frac{1}{x-z} d\mu$$

Theorem

(Liu-Pego-Donoghue, 2014) The measure can be recovered by

$$\mu(\alpha, \beta) = -\lim_{y \to 0^+} \frac{1}{\pi} \int_\alpha^\beta \Im(f(x+iy))dx.$$

Hilbert Matrix Example Continued

$F(z) = -\log(1-z)$, and

$$-f(z) = \log(1-1/z) = \log\left(\frac{x^2+y^2-x+iy}{x^2+y^2}\right)$$

As $y \to^{+}$,

$$-f(z) \to \log\left(\frac{x^2-x}{x^2}\right) = \log\left(\frac{x-1}{x}\right)$$

Hence

$$-\Im(f(z)) \to \pi$$

and

$$\mu(\alpha,\beta) = \beta - \alpha, \qquad \text{i.e., } \mu \text{ is Lebesgue measure on } [0,1]$$

A Second Example

- Take $F(z) = e^z - 1 = \sum_{k=1}^{\infty}(1/k!)z^k$
- Then $f(z) = e^{1/z} - 1$ is analytic outside $E := \{0\}$
- Polya's Theorem implies that
- $$\lim_{n\to\infty} |\mathrm{det}(H_n)|^{1/n^2} = \mathrm{tfd}(E) = 0$$
- But it is possible to calculate

$$|\mathrm{det}(H_n)| = \left(\prod_{j=1}^{n-1} j!\right) \Big/ \left(\prod_{j=1}^{2n-1} j!\right)$$

- So that

$$\lim_{n\to\infty} |\mathrm{det}(H_n)|^{1/(n^2\log(n))} = e^{-1}$$

An Example Sarting From the Measure μ

- Take $d\mu = \frac{2}{\pi}\sqrt{\frac{1-x}{x}}dx \in \mathrm{Reg}([0,1])$
- Calculate

$$F(z) = \int_0^1 \frac{z}{1-xz}d\mu(x) = 2(1-\sqrt{1-z})$$

- Bloom's Theorem implies that
- $$\lim_{n\to\infty} |\det(H_n)|^{1/n^2} = \mathrm{tfd}([0,1]) = 1/4$$

- But it is also possible to calculate explicitly

$$|\det(H_n)| = 4^{-n(n-1)}$$

Recall the Determinant of T

Recall that our original reason was to study

$$\det(T) = \begin{vmatrix} q_2 & q_3 & \cdot & \cdot & q_{n+1} \\ q_3 & q_4 & \cdot & \cdot & q_{n+2} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ q_{n+1} & q_{n+2} & \cdot & \cdot & q_{2n} \end{vmatrix}$$

But there is a shift – it begins with q_2 and not q_1

How to Handle a Shift

- $f(z) = \int_0^1 \frac{1}{z-x} d\mu(x) = \frac{a_1}{z} + \frac{a_2}{z} + \cdots$
- $$\begin{aligned} f_1(z) &:= zf(z) - a_1 = \frac{a_2}{z} + \frac{a_3}{z} + \cdots \\ &= z\int_0^1 \frac{1}{z-x} d\mu(x) - a_1 \\ &= \int_0^1 \frac{1}{z-x} x d\mu(x) \end{aligned}$$

Shifted Measure

Lemma

If $\mu \in \mathrm{Reg}([0,1])$ *then so is* $x^r\mu$, $r = 0, 1, 2, \cdots$.

Corollary

If $\mu \in \mathrm{Reg}([0,1])$ *then*

$$\lim_{n\to\infty} |\mathrm{det}(H_n(f_r))|^{1/n^2} = \mathrm{tfd}([0,1]) = \frac{1}{4}$$

Scaling

- Want to consider even a simple quadratic $y^2 + x = 1$, i.e., $y = \sqrt{1-x}$
- But radius of convergence is one
- Need to scale, i.e., $y = \sqrt{R-x} = \sqrt{R}\sqrt{1-x/R}$
- Consider $F(z/R) = \sum_{k=1}^{\infty} a_k R^{-k} z^k$.
- Then, e.g.,

$$H_3(F(z/R)) = \begin{bmatrix} a_1/R & a_2/R^2 & a_3/R^3 \\ a_2/R^2 & a_3/R^3 & a_4/R^4 \\ a_3/R^3 & a_4/R^4 & a_5/R^5 \end{bmatrix}$$

Calculate the Scaled Determinant

Factor out R^{-j} from column j to get

$$
\begin{aligned}
&R^{-1}R^{-2}R^{-3}\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2/R^1 & a_3/R^1 & a_4/R^1 \\ a_3/R^2 & a_4/R^2 & a_5/R^2 \end{vmatrix} \\
&= R^{-6}R^{-1}R^{-2}\mathrm{det}(H_3(F(z))) \\
&= R^{-9}\mathrm{det}(H_3(F(z)))
\end{aligned}
$$

In general get $R^{-n^2}\mathrm{det}(H_n(F(z)))$
so that the limit of the $1/n^2$ power gets multiplied by R^{-1}.

Conclusion

Theorem

For $A(z) = \sqrt{R - z}$, $R > 1$,*and* $Q_{2n}(z)$ *its interpolant at the* $2n + 1$*th roots of unity,*

$$\lim_{n\to\infty} |\det(H_n^{(1)}(Q_{2n}))|^{1/n^2} = \frac{1}{4R}.$$

Thank you for your attention!